1. (a) Suppose that \( f \) is affine. For \( x \in X \), we may determine \( f^{-1}(x) \) by pulling back along the canonical map \( \text{Spec} \kappa(x) \to X \). We may thus assume that \( X = \text{Spec}(k) \) for \( k \) a field. In this case, problem 8 of Math 203B PS 8 asserts that \( f \) is finite, so in particular it has finite fibers.

Suppose that \( f \) has finite fibers. For any \( x \in X \), \( f^{-1}(x) \) is a finite subset of \( \mathbb{P}^n_{\kappa(x)} \), so we can find a hypersurface in \( \mathbb{P}^n_{\kappa(x)} \) disjoint from this subset. (Proof: let \( Z \) be the reduced closed subscheme of \( \mathbb{P}^n_{\kappa(x)} \) with underlying set \( f^{-1}(x) \). For \( d \) large, \( \Gamma(\mathbb{P}^n_{\kappa(x)}, \mathcal{O}(d)) \) surjects onto \( \Gamma(Z, \mathcal{O}(d)) \cong \Gamma(Z, \mathcal{O}) \), so we may lift the constant function \( 1 \in \Gamma(Z, \mathcal{O}) \) to \( \Gamma(\mathbb{P}^n_{\kappa(x)}, \mathcal{O}(d)) \). This defines a suitable hypersurface.) This lifts to a hypersurface \( H \) in \( \mathbb{P}^n_U \) for some open affine neighborhood \( U \) of \( x \) in \( X \). But \( Z = \mathbb{P}^n_U \setminus H \) is affine, so \( f|_U \) factors through a closed immersion \( Y \times_X U \to Z \). Hence \( f \) is affine.

(b) We may assume \( X = \text{Spec}(R) \) is affine. Since \( f \) is affine, \( Y = \text{Spec}(S) \) for \( S = \Gamma(Y, \mathcal{O}_Y) \cong \Gamma(X, f_*\mathcal{O}_Y) \). But since \( f \) is projective and \( X \) is noetherian, \( \Gamma(X, f_*\mathcal{O}_Y) \) is a finite \( \mathcal{O}_Y \)-module. Hence \( S \) is a finite \( R \)-algebra, proving the claim.

2. We describe two different constructions. The first construction is to recall that \( M \), being finite projective, is locally free, and that the usual trace on a square matrix is invariant under conjugation. Consequently, the local trace pieces together to give a well-defined section of the structure sheaf on \( R \), and hence an element of \( R \).

The second construction is to choose a free module \( F \) admitting a direct sum decomposition \( M \oplus N \) for some \( N \). For \( T \in \text{Hom}_R(M, M) \), we may then set \( \text{Trace}(T, M) = \text{Trace}(T \oplus 0, M \oplus N) \). To see that this does not depend on any choices, note that adding a free summand to \( N \) clearly has no effect. So if \( M \oplus N' \cong F' \) is another isomorphism, then

\[
\text{Trace}(T \oplus 0, M \oplus N) = \text{Trace}(T \oplus 0 \oplus 0 \oplus 0, M \oplus N \oplus M \oplus N') = \text{Trace}(T \oplus 0 \oplus 0 \oplus 0, M \oplus N' \oplus M \oplus N) = \text{Trace}(T \oplus 0, M \oplus N').
\]

3. Note that the universal property need only be checked in the case where \( X'_0 \) is defined by an ideal \( I \) of \( R \) with square zero.

(a) We prove locality on the target, the argument for locality on the source being similar. In one direction, if \( U \subseteq X \) is an open subscheme, then we can test the formally ramified property for \( Y \times_X U \to U \) with the original diagram, by considering only maps \( X'_0 \to Y \) factoring through \( Y \times_X U \). In the other direction, if \( \{U_i\}_{i \in I} \) is an open covering of \( X \), we can test the formally unramified property by restricting to each \( Y \times_X U_i \) and glueing maps together.
(b) By (a), both properties are local on the source and target, so we may assume that $X = \text{Spec}(R)$ and $Y = \text{Spec}(S)$ are both affine. Take $X' = \text{Spec}(R')$. For $I$ an ideal of $R'$ with square zero, given two $R$-algebra homomorphisms $f_1, f_2 : S \to R'$, we get a derivation $d : S \to I$ by mapping $s$ to $f_1(s) - f_2(s)$. If $\Omega_{S/R} = 0$, then $d$ must be zero, so $f_1 = f_2$ and $f$ is formally unramified. Conversely, if $\Omega_{S/R} \neq 0$, we may take $R' = S \oplus \Omega_{S/R}$ and the two maps $s \mapsto s \oplus 0$ and $s \mapsto s \oplus ds$ to get a counterexample against the formally unramified property.

4. Locality on the target is a formal consequence of locality on the source (because open immersions are formally étale), so we focus on the latter. By affine communication, we may assume that $Y = \text{Spec}(S)$ is affine and covered by distinguished open subsets $D(g_i)$ which are formally smooth over $X$. Use $X'_0 \to Y$ to pull back $g_i$ to $R'/I$, then lift to some $\tilde{g}_i \in R'$. As in part (b) of the previous exercise, any two liftings $X'_{\tilde{g}_i} \to Y_{g_i}$ differ by an element of $\text{Hom}_{S_{R'_i}}(\Omega_{S_{R'_i}/R_{g_i}}, I_{\tilde{g}_i})$; we thus get a 1-cocycle for the quasicoherent sheaf $\text{Hom}_{S}(\Omega_{S/R}, I)$ on the affine scheme $\text{Spec}(S)$. Therefore it is also a coboundary, and we get a global lifting.

5. Since flatness is local on the source and target, this reduces to a statement about rings: if $R \to S$ is a ring homomorphism, $R \to T$ is a faithfully flat ring homomorphism, and $T \to S \times_R T$ is flat, then $R \to S$ is flat. To check this, let $M \to N$ be an injective morphism of $R$-modules. Then $M \otimes_R T \to N \otimes_R T$ is injective, as then is $M \otimes_R (S \otimes_R T) \to N \otimes_R (S \otimes_R T)$. Since $R \to T$ is faithfully flat, this implies that $M \otimes_R S \to N \otimes_R S$ is flat.

6. Suppose the Jacobian condition is satisfied. It is then clear that the morphism is of finite presentation. Let $R'$ be a local $R$-algebra, let $I$ be an ideal of $R'$ of square zero, and let $S \to R'/I$ be an $R$-algebra homomorphism; we must exhibit a lifted homomorphism $S \to R'$. Let $\overline{y}_1, \ldots, \overline{y}_n$ be the images of $x_1, \ldots, x_n \in R'/I$; we must lift these to $y_1, \ldots, y_n \in R'$ so that $f_i(y_1, \ldots, y_n) = 0$ for $i = 1, \ldots, m$. If we start with arbitrary lifts $z_1, \ldots, z_n$, instead, we must then solve the equations

$$0 = f_i(z_1 + \delta_1, \ldots, z_n + \delta_n) \quad (i = 1, \ldots, n)$$

for $i = 1, \ldots, m$ with $\delta_1, \ldots, \delta_n \in I$. But since $I$ is of square zero,

$$0 = f_i(z_1, \ldots, z_n) + \sum_{j=1}^n \delta_j \frac{\partial f_i}{\partial x_j}(z_1, \ldots, z_n).$$

Over the residue field of $R'$, the Jacobian criterion guarantees that we can do linear algebra to solve for the $\delta_j$; the same is then true in $R'/I$ because $R'$ is a local ring. It follows that $\text{Spec}(S) \to \text{Spec}(R)$ is formally smooth.

7. The $S$-module $\Omega_{S/R}$ is generated by elements of the form $ds$ with $s \in S$. However, by hypothesis each $s \in S$ has the form $t^p$ for some $t \in S$, and $ds = pt^{p-1} dt = 0$ because $S$ is of characteristic $p$. 

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