Math 203C (Algebraic Geometry), UCSD, spring 2013
Solutions for problem set 1

1. (a) Suppose that f is affine. For $x \in X$, we may determine $f^{-1}(x)$ by pulling back along the canonical map $\operatorname{Spec} \kappa(x) \to X$. We may thus assume that $X = \operatorname{Spec}(k)$ for k a field. In this case, problem 8 of Math 203B PS 8 asserts that f is finite, so in particular it has finite fibers.

   Suppose that f has finite fibers. For any $x \in X$, $f^{-1}(x)$ is a finite subset of $\mathbb{P}^n_{\kappa(x)}$, so we can find a hypersurface in $\mathbb{P}^n_{\kappa(x)}$ disjoint from this subset. (Proof: let Z be the reduced closed subscheme of $\mathbb{P}^n_{\kappa(x)}$ with underlying set $f^{-1}(x)$. For d large, $\Gamma(\mathbb{P}^n_{\kappa(x)}, \mathcal{O}(d))$ surjects onto $\Gamma(Z, \mathcal{O}(d)) \cong \Gamma(Z, \mathcal{O})$, so we may lift the constant function $1 \in \Gamma(Z, \mathcal{O})$ to $\Gamma(\mathbb{P}^n_{\kappa(x)}, \mathcal{O}(d))$. This defines a suitable hypersurface.) This lifts to a hypersurface H in $\mathbb{P}^n_U$ for some open affine neighborhood U of x in X. But $Z = \mathbb{P}^n_U \setminus H$ is affine, so $f|_U$ factors through a closed immersion $Y \times_X U \to Z$. Hence f is affine.

   (b) We may assume $X = \operatorname{Spec}(R)$ is affine. Since f is affine, $Y = \operatorname{Spec}(S)$ for $S = \Gamma(Y, \mathcal{O}_Y) \cong \Gamma(X, f_* \mathcal{O}_Y)$. But since f is projective and X is noetherian, $\Gamma(X, f_* \mathcal{O}_Y)$ is a finite $\mathcal{O}_Y$-module. Hence S is a finite R-algebra, proving the claim.

2. We describe two different constructions. The first construction is to recall that M, being finite projective, is locally free, and that the usual trace on a square matrix is invariant under conjugation. Consequently, the local trace pieces together to give a well-defined section of the structure sheaf on R, and hence an element of R.

   The second construction is to choose a free module F admitting a direct sum decomposition $M \oplus N$ for some N. For $T \in \operatorname{Hom}_R(M, M)$, we may then set $\operatorname{Trace}(T, M) = \operatorname{Trace}(T \oplus 0, M \oplus N)$. To see that this does not depend on any choices, note that adding a free summand to N clearly has no effect. So if $M \oplus N' \cong F'$ is another isomorphism, then

$$\begin{aligned}\operatorname{Trace}(T \oplus 0, M \oplus N) &= \operatorname{Trace}(T \oplus 0 \oplus 0 \oplus 0, M \oplus N \oplus M \oplus N') \\ &= \operatorname{Trace}(T \oplus 0 \oplus 0 \oplus 0, M \oplus N' \oplus M \oplus N) \\ &= \operatorname{Trace}(T \oplus 0, M \oplus N').\end{aligned}$$

3. Note that the universal property need only be checked in the case where X'_0 is defined by an ideal I of R with square zero.

   (a) We prove locality on the target, the argument for locality on the source being similar. In one direction, if $U \subseteq X$ is an open subscheme, then we can test the formally ramified property for $Y \times_X U \to U$ with the original diagram, by considering only maps $X'_0 \to Y$ factoring through $Y \times_X U$. In the other direction, if $\{U_i\}_{i \in I}$ is an open covering of X, we can test the formally unramified property by restricting to each $Y \times_X U_i$ and glueing maps together.

(b) By (a), both properties are local on the source and target, so we may assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ are both affine. Take $X' = \mathrm{Spec}(R')$. For I an ideal of R' with square zero, given two R-algebra homomorphisms $f_1, f_2 : S \to R'$, we get a derivation $d : S \to I$ by mapping s to $f_1(s) - f_2(s)$. If $\Omega_{S/R} = 0$, then d must be zero, so $f_1 = f_2$ and f is formally unramified. Conversely, if $\Omega_{S/R} \neq 0$, we may take $R' = S \oplus \Omega_{S/R}$ and the two maps $s \mapsto s \oplus 0$ and $s \mapsto s \oplus ds$ to get a counterexample against the formally unramified property.

4. Locality on the target is a formal consequence of locality on the source (because open immersions are formally étale), so we focus on the latter. By affine communication, we may assume that $Y = \mathrm{Spec}(S)$ is affine and covered by distinguished open subsets $D(g_i)$ which are formally smooth over X. Use $X'_0 \to Y$ to pull back g_i to R'/I, then lift to some $\tilde{g}_i \in R'$. As in part (b) of the previous exercise, any two liftings $X'_{\tilde{g}_i} \to Y_{g_i}$ differ by an element of $\mathrm{Hom}_{S_{\tilde{g}_i}}(\Omega_{S_{\tilde{g}_i}/R_{\tilde{g}_i}}, I_{\tilde{g}_i})$; we thus get a 1-cocycle for the quasicoherent sheaf $\mathrm{Hom}_S(\Omega_{S/R}, I)$ on the affine scheme $\mathrm{Spec}(S)$. Therefore it is also a coboundary, and we get a global lifting.

5. Since flatness is local on the source and target, this reduces to a statement about rings: if $R \to S$ is a ring homomorphism, $R \to T$ is a faithfully flat ring homomorphism, and $T \to S \times_R T$ is flat, then $R \to S$ is flat. To check this, let $M \to N$ be an injective morphism of R-modules. Then $M \otimes_R T \to N \otimes_R T$ is injective, as then is $M \otimes_R (S \otimes_R T) \to N \otimes_R (S \otimes_R T)$. Since $R \to T$ is faithfully flat, this implies that $M \otimes_R S \to N \otimes_R S$ is flat.

6. Suppose the Jacobian condition is satisfied. It is then clear that the morphism is of finite presentation. Let R' be a local R-algebra, let I be an ideal of R' of square zero, and let $S \to R'/I$ be an R-algebra homomorphism; we must exhibit a lifted homomorphism $S \to R'$. Let $\overline{y}_1, \dots, \overline{y}_n$ be the images of $x_1, \dots, x_n \in R'/I$; we must lift these to $y_1, \dots, y_n \in R'$ so that $f_i(y_1, \dots, y_n) = 0$ for $i = 1, \dots, m$. If we start with arbitrary lifts $z_1, \dots, z_n$ instead, we must then solve the equations

$$0 = f_i(z_1 + \delta_1, \dots, z_n + \delta_n) \qquad (i = 1, \dots, n)$$

for $i = 1, \dots, m$ with $\delta_1, \dots, \delta_n \in I$. But since I is of square zero,

$$0 = f_i(z_1, \dots, z_n) + \sum_{j=1}^{n} \delta_j \frac{\partial f_i}{\partial x_j}(z_1, \dots, z_n).$$

Over the residue field of R', the Jacobian criterion guarantees that we can do linear algebra to solve for the δ_j; the same is then true in R'/I because R' is a local ring. It follows that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is formally smooth.

7. The S-module $\Omega_{S/R}$ is generated by elements of the form ds with $s \in S$. However, by hypothesis each $s \in S$ has the form t^p for some $t \in S$, and $ds = pt^{p-1}\, dt = 0$ because S is of characteristic p.