Math 203C (Algebraic Geometry), UCSD, spring 2013
Problem Set 1 (due Wednesday, April 10)

Solve the following problems, and turn in the solutions to *four* of them. As usual, please document any collaboration and cite all external references. These now include results from Math 203B problem sets.

The first lecture will be Monday, April 8, at 11:00 AM in APM 7421. See you then!

1. Let $f : Y \to X$ be a proper morphism of schemes.
   (a) Prove that f is affine if and only if f has finite fibers. Hint: for one direction, look at 203B problem sets; for the other direction, remember that every finite subset of $\mathbb{P}^n_k$ for k a field lies in the complement of some hypersurface.
   (b) Assume that X is noetherian and f is projective (i.e., the composition of a closed immersion $Y \to \mathbb{P}^n_X$ with the projection $\mathbb{P}^n_X \to X$). Prove that if f is affine, then it is finite. Hint: use Zariski's main theorem.
   (c) Optional: eliminate the extra hypotheses in (b).

2. Prove that for every ring R and every finite projective R-module M, there is an R-linear map Trace : $\mathrm{Hom}_R(M, M) \to R$ which is functorial in both R and M and which computes the usual trace (the sum of diagonal entries) when M is a finite free R-module. You don't have to be too careful about checking the functoriality as long as your construction is correct, but here's what I mean: functoriality in R means that if $f : R \to S$ is a homomorphism, then for any $g \in \mathrm{Hom}_R(M, M)$, the trace of $g \otimes 1 \in \mathrm{Hom}_S(M \otimes_R S, M \otimes_R S)$ equals $f(\mathrm{Trace}(g))$. Functoriality in M means that if

$$0 \to M_1 \to M \to M_2 \to 0$$

is a short exact sequence and T is an endomorphism of M which induces endomorphisms T_1, T_2 on M_1, M_2, then $\mathrm{Trace}(T_1) + \mathrm{Trace}(T_2) = \mathrm{Trace}(T)$.

3. Let $f : Y \to X$ be a morphism of schemes. Recall that f is defined to be *formally smooth/unramified/étale* if for every affine scheme $X' = \mathrm{Spec}(R)$ and every closed subscheme X'_0 of X defined by a nilpotent ideal of R, every diagram

$$\begin{tikzcd} X'_0 \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

admits at least/at most/exactly one extension by a dashed arrow. Following EGA, we say that f is *smooth/unramified/étale* if f is formally smooth/unramified/étale and locally of finite presentation (the Stacks Project uses a different but ultimately equivalent characterization using the naïve cotangent complex).

(a) Prove that the property of being formally unramified is local on both the source and the target.

(b) Prove that f is formally unramified if and only if $\Omega_{Y/X} = 0$.

4. Prove that the property of being formally smooth is local on both the source and the target. Hint: the hard part is locality on the target; follow (b) of the previous problem to reduce to the fact that any 1-cocycle of a quasicoherent sheaf on an affine scheme is trivial.

5. Let $Y \to X$ be a morphism and let $Z \to X$ be a faithfully flat (flat and surjective) morphism such that $Y \times_X Z \to Z$ is flat. Prove that $Y \to X$ is flat. This is a simple example of *faithfully flat descent*, more on which later.

6. Let S be a finitely presented R-algebra. We say that S satisfies the *Jacobian criterion* if $S \cong R[x_1, \dots, x_n]/(f_1, \dots, f_m)$ for some $0 \leq m \leq n$ such that the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial x_j} \right)$$

has rank $n - m$ everywhere (i.e., the $m \times m$ subdeterminants generate the unit ideal). Prove that if S satisfies the Jacobian criterion, then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is smooth. You may use without proof the fact that it suffices to check the formally smooth property when X' is the spectrum of a local ring (this amounts to the fact that the formally smooth property is local on the source).

7. Let $R \to S$ be a homomorphism of rings of characteristic $p > 0$. Suppose that the Frobenius homomorphism $x \to x^p$ on S is surjective. Prove that $\Omega_{S/R} = 0$.