Math 203C (Algebraic Geometry), UCSD, spring 2013
Solutions for problem set 2

1. (a) We first check that $\Omega_{S/R}$ is free on the generator dx. On one hand, it is clear that dx generates $\Omega_{S/R}$ since S is a quotient of $S = \mathbb{F}_p[x]$. On the other hand, we have a well-defined R-linear derivation $S \to S\,dx$ given by $\sum_{i=0}^{p-1} c_i x^i \mapsto \sum_{i=0}^{p-2} i c_i x^{i-1}\,dx$, which induces an isomorphism $\Omega_{S/R} \cong S\,dx$.

   We now check that $R \to S$ is not formally smooth. We must produce an R-algebra T, an ideal I of square zero, and an R-algebra homomorphism $S \to T/I$ which does not factor through T. We take $T = \mathbb{F}_p[x^{2p}]$ and $I = x^p T$; then $S \cong T/I$ but $x \in T$ does not lift to an element whose p-th power is zero.

   (b) Put $R = \mathbb{F}_p(x^p)$ and $S = \mathbb{F}_p(x)$. Then again, $\Omega_{S/R}$ is generated by x since x generates S as an R-algebra. On the other hand, we have a well-defined R-linear derivation $S \to S\,dx$ given by $\sum_{i=0}^{p-1} c_i x^i \mapsto \sum_{i=0}^{p-2} i c_i x^{i-1}\,dx$, which induces an isomorphism $\Omega_{S/R} \cong S\,dx$.

2. It is obvious that $k \to \ell$ is finitely presented, so it remains to check that $k \to \ell$ is formally étale. By the primitive element theorem, we can write $\ell = k[t]/(P(t))$ for some monic polynomial $P \in k[t]$ which is separable (i.e., over an algebraic closure of k it factors into *distinct* linear factors). For any ring homomorphism $\ell \to R/I$ in which R is a ring and I is an ideal of square zero, we must check that there is a unique factorization $\ell \to R \to R/I$; that is, there is a unique root x of $P(t)$ in R lifting the image y of t in R/I. To see this, let x_0 be any lift of y; then we can and must take

$$x = x_0 - P(x_0)/P'(x_0).$$

(Note that $P'(x_0)$ reduces to a unit in R/I and hence is a unit in R, because the nilradical of R is contained in the Jacobson radical.) Therefore $k \to \ell$ is formally étale and hence étale; that is, (iii) implies (i).

3. Since $R \to S$ is unramified, $\Omega_{S/R} = 0$. Let I be the kernel of $S \otimes_R S \to S$; then $I/I^2 \cong \Omega_{S/R} = 0$, so $I = I^2$. Since I is the kernel of a map from a finitely generated R-algebra to a finitely presented R-algebra, it is a finitely generated ideal (we will show this on a later problem set). By Nakayama's lemma, we must have $I_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \operatorname{Spec}(S)$. But the set of $x \in \operatorname{Spec}(S) \times_{\operatorname{Spec}(R)} \times \operatorname{Spec}(S)$ at which $I_x = 0$ is open and contained in the image of Δ, so Δ is an open immersion as claimed. (Taken from EGA 4, 16.1.9.)

4. (a) Let x be a closed point of $X = \operatorname{Spec}(S)$. Since k is algebraically closed, by the Nullstellensatz we must have $\kappa(x) = k$ and hence there is a natural section $s : \operatorname{Spec}(k) = \operatorname{Spec}(\kappa(x)) \to X$ of the structure map $X \to \operatorname{Spec}(k)$. But $\Delta(X)$ is open in $X \times_{\operatorname{Spec}(k)} X$ by the previous problem, and the inverse image of $\Delta(X)$ under $s \times 1_X : X \to X \times_{\operatorname{Spec}(k)} X$ is equal to $\{x\}$. Hence $\{x\}$ is an open set, as claimed.

(b) By (a), the closed points of $\mathrm{Spec}(S)$ are isolated, so S must be a finite k-algebra. We may assume that $\mathrm{Spec}(S)$ is a single point, whose residue field must equal k since k is algebraically closed. But then Δ is both an open immersion and a bijection on points, hence an isomorphism; that is, $S \otimes_k S \cong S$ and so $\dim_k(S) = \dim_k(S)^2$. This forces $S = k$. (Taken from EGA 4, 17.4.1.)

5. Let ℓ be an algebraic closure of k. By the previous problem, $S \otimes_k \ell$ is a direct sum of finitely many copies of k, and in particular is reduced. Since S is automatically flat over k, S is also reduced. Since $\dim_k S = \dim_\ell(S \otimes_k \ell)$, S is a finite k-algebra and hence an Artinian k-algebra. It thus splits into finitely many connected components, so we may reduce to the case where S is connected. Since it is also reduced, it is a field extension of k of finite degree. Since $S \times_k \ell$ is a direct sum of $[S : k]$ copies of ℓ, S admits $[S : k]$ distinct embeddings into ℓ, so it must be a separable field extension.

6. Let $x \in X, y \in Y$ be points with $f(y) = x$. Since $Y \times_X \kappa(x) \to \kappa(x)$ is unramified, by an earlier problem $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a finite separable field extension of $\kappa(x)$. By the primitive element theorem, this field extension has the form $\kappa(x)[t]/(P(t))$ for some irreducible polynomial $P(t)$. By lifting this polynomial, we see that $\mathcal{O}_{Y,y}$ satisfies the Jacobian criterion and is thus smooth over $\mathcal{O}_{X,x}$. Hence f is smooth and unramified, hence étale.

7. (a) If we let y be a coordinate for the bottom $\mathbb{P}^1_k$, then we are adjoining a root x of $x^p + x^{-1} = y$, or equivalently $x^{p+1} - xy + 1 = 0$. For y finite, it is clear that we get a finite flat ring extension of degree $p + 1$. The extension is also unramified because $d(x^p + x^{-1}) = d(x^{-1})$ has no zero or pole away from 0 and ∞, and only those points lie over ∞.

   (b) It is not possible for $k = \mathbb{C}$ because $\mathbb{P}^1_k - \{\infty\}$ is simply connected, and so admits no connected covering space of degree greater than 1.