**Math 203C (Algebraic Geometry), UCSD, spring 2013**
**Problem Set 2 (due Wednesday, April 17)**

Solve the following problems, and turn in the solutions to *four* of them.

1. (a) It was shown in class that if $R \to S$ is a formally smooth morphism of rings, then $\Omega_{S/R}$ is a projective $S$-module. Prove that the converse fails using the example $R = \mathbb{F}_p$, $S = \mathbb{F}_p[x]/(x^p)$.

   (b) Similarly, show that the purely inseparable field extension $\mathbb{F}_p(x) \to \mathbb{F}_p(x^p)$ is not formally unramified by computing $\Omega_{\mathbb{F}_p(x)/\mathbb{F}_p(x^p)}$ and seeing that it is nonzero.

2. Let $k$ be a field. Let $\ell$ be a finite separable field extension of $k$ of finite degree. Prove that the morphism $k \to \ell$ is étale.

3. Let $R \to S$ be an unramified ring morphism. Prove that $\Delta : \mathrm{Spec}(S) \to \mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ is an open immersion. Hint: put $I = \ker(S \otimes_R S \to S)$ and identify points at which the stalk of $I$ vanishes. You may assume that $I$ is a finitely generated ideal of $S \otimes_R S$; we'll prove this on a later problem set.

4. Let $k$ be an algebraically closed field and let $k \to S$ be an unramified ring morphism.

   (a) Prove that every closed point of $X = \mathrm{Spec}(S)$ is isolated. Hint: construct a map $X \to X \times_{\mathrm{Spec}(k)} X$ whose image meets the diagonal at a single point.

   (b) Prove that $S$ is a finite direct sum of copies of $k$.

5. Let $k$ be a field and let $k \to S$ be an unramified ring homomorphism. Prove that $S$ is a direct sum of finitely many separable field extensions of $k$; in particular, $k \to S$ is finite étale.

6. Let $f : Y \to X$ be a morphism of schemes. It was shown in class that if $f$ is étale, then $f$ is flat and unramified. Prove conversely that if $f$ is flat and unramified, then $f$ is étale. Hint: use the primitive element theorem plus the Jacobian criterion for smoothness.

7. (a) Let $k$ be a field of characteristic $p > 0$. Prove that the map $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $x \mapsto x^p + x^{-1}$ is finite étale of degree $p + 1$ over the complement of $\infty$.

   (b) From part (a), it follows that $\mathbb{P}^1_k - \{0, \infty\}$ can be written as a finite étale cover of $\mathbb{P}^1_k - \{\infty\}$. Is this possible for $k = \mathbb{C}$?