Math 203C (Algebraic Geometry), UCSD, spring 2013
Solutions for problem set 3

1. (a) Let K be the kernel of f. By hypothesis, there is another surjection $f' : F' \to M$ with F' a finite free R-module such that $\ker(f')$ is finitely generated. Put $F'' = F \oplus F'$ and $f'' = f \oplus f' : F \oplus F' \to M$. Since F' is a free module, we may factor f' through a map $g' : F' \to F$ to obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K'' & \longrightarrow & F'' & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f \oplus g'} & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

whose second and third columns are exact. By the snake lemma or the five lemma, $K'' \to K$ is finitely generated. But K'' is a finitely generated R-module: it is enough to take $0 \oplus m'$ for m' running over a finite generating set for $\ker(f')$ together with $m \oplus m'$ for m running over a finite generating set for F and m' a lift of $-f(m)$ to F'. Therefore K is also finitely generated.

   (b) *Mutatis mutandis.*

   (c) Write $S = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$ as an R-algebra and choose $g_1, \dots, g_k \in R[x_1, \dots, x_n]$ whose images in S generate S as an R-module. For $i = 1, \dots, n$, choose $h_{ij} \in R$ and $p_{ij} \in R[x_1, \dots, x_n]$ such that

$$x_i = \sum_{j=1}^{k} h_{ij} g_j + \sum_{j=1}^{m} p_{ij} f_j.$$

Let R' be the subring of R generated over $\mathbb{Z}$ by everything in sight (that is, the elements h_{ij} and the coefficients of the polynomials f_i, g_j, p_{ij}); then R' is a finitely generated $\mathbb{Z}$-algebra and hence a noetherian ring by the Hilbert basis theorem. Put $S' = R'[x_1, \dots, x_n]/(f_1, \dots, f_m)$; then S' is again generated by the x_i as an R'-module. But R' is noetherian, so S' must be not only finitely generated but finitely presented as an R'-module. This proves the claim.

2. To prove that (i) implies (ii), recall that any étale morphism is smooth (obvious from the definitions) and any smooth morphism is flat (previous exercise). So if $R \to S$ is finite étale, it is finite flat and also finitely presented as an algebra map. By the previous exercise, S is also finitely presented as an R-module; but a finitely presented flat R-module is projective. So S is projective as a module over R. On the other hand, if $R \to S$ is unramified, then by a previous exercise, $\operatorname{Spec}(S) \to \operatorname{Spec}(S \otimes_R S)$ is an open immersion. But it's also a closed immersion because $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is separated (it's an affine morphism!), so $\operatorname{Spec}(S \otimes_R S)$ splits as the disjoint union of $\operatorname{Spec}(S)$ and another subscheme. That is, $S \otimes_R S$ splits as the direct sum of S and another subring, so $S \otimes_R S \to S$ is finite projective.

To prove that (ii) implies (i), note that S being finite projective as an R-module implies that it is finitely presented and flat. In particular, S is finitely presented as an R-algebra, since any module generators and relations can also be used as algebra generators and relations. On the other hand, S being finite projective over $S \otimes_R S$ implies that $R \to S$ is formally unramified: if $I = \ker(S \otimes_R S \to S)$, then we have an isomorphism $S \otimes_R S \cong S \oplus I$ of $S \otimes_R S$-modules and hence of rings, meaning $\Omega_{S/R} = I/I^2 = 0$. So $R \to S$ is finitely presented, flat, and formally unramified, hence flat and unramified, hence étale by a previous exercise.

To prove that (i) and (iii) are equivalent, we may assume from the outset that S is finite projective as an R-module. Then both conditions can be checked on local rings, so we may assume further that R is local with maximal ideal I. In this case, S becomes a finite free R-module.

To prove that (i) implies (iii), we first check it when $I = 0$: it is then harmless to pass from $k = R$ to an algebraic closure, in which case S becomes a direct sum of copies of k and the claim is clear. In general, by Nakayama's lemma it follows that $S \to \mathrm{Hom}_R(S, R)$ is surjective, but these are two free modules of the same rank! So this map must be an isomorphism. (Namely, composing it with an isomorphism of free modules gives a surjective endomorphism of a finite free module, whose determinant must then be a unit. So this endomorphism must be invertible.)

To prove that (iii) implies (i), we need only check that $R \to S$ is formally unramified. We first check this when $I = 0$, so $k = R$ is a field and S is an artinian semilocal ring. We may assume S is itself local. Since computing $\Omega_{S/R}$ commutes with extending the base field, we may assume k is algebraically closed. Note that any nilpotent $x \in S$ has trace zero because multiplication by x defines a nilpotent matrix in $\mathrm{Hom}_R(S, S)$, and nilpotent matrices have all eigenvalues equal to 0. Therefore S must be reduced; since k is algebraically closed, we must have $S = k$. This proves the claim when k is a field.

For general R, we know from the field case that $\Omega_{S/R} \otimes_R (R/I) = 0$. Since $\Omega_{S/R}$ is a finite R-module, Nakayama's lemma implies that it is zero.

3. Let I, J be the maximal ideals of R, S. Suppose first that $R \to S$ is étale. Then $R/I \to S/I$ is étale, so it is reduced (proving (i)) and a direct sum of separable field extensions (proving (i)).

   Conversely, suppose (i) and (ii) hold. Then S is generated (freely) over R by the lifts of any basis of S/J over R/I. By the primitive element theorem we can find an element $s \in S$ whose minimal polynomial $Q(t)$ over K has coefficients in R and reduces to a separable polynomial over R/I. But then $S \cong R[t]/(Q(t))$ and $Q'(t)$ is invertible in S (because it has nonzero image in S/J), so $\Omega_{S/R} = 0$. Therefore $R \to S$ is formally unramified; since it's also finite flat (any finite torsion-free module over a DVR is free and hence flat), it is étale.

4. (a) There are many ways to distinguish X and Y. For example, any two irreducible closed subschemes of X which are not points have nonempty intersection, whereas

this is not true for Y (take the preimages of two distinct points in $\mathbb{P}^1_k$ under one of the projection maps).

(b) We take Z to be the blowup of X at $[0:0:1]$. Then the family of lines in question is given by the Zariski closures of the lines $y = mx$ for $m \in k$ (i.e., the proper transforms of the lines through $[0:0:1]$).

(c) (This solution incorporates the correction: the blowup of Y is at one point, not two.) The motivation is given by (b): if we blow up X at any two points, we produce two distinct families of pairwise disjoint lines, which we might expect to be the proper transforms of the horizontal and vertical lines in $\mathbb{P}^1_k$.

In order to transform this into a formal construction, let us step through the geometry more carefully. Let us take the two points of X to be $[1:0:0]$ and $[0:1:0]$ (in the xy-plane, these are the vertical and horizontal points at infinity). The lines through $[1:0:0]$ can be distinguished by their values of $[y:z]$, while the lines through $[0:1:0]$ may be distinguished by their values of $[x:z]$. This leads us to define a rational map from X to Y by

$$[x:y:z] \mapsto ([y:z],[x:z]),$$

which is defined on the complement of the two points $[1:0:0],[0:1:0]$. Let $\tilde{X}$ be the Zariski closure of the graph of this rational map; it is easily checked to be the blowup of X at the two missing points.

In the other direction, we may consider the rational map

$$([a:b],[c:d]) \mapsto [c/d : a/b : 1] = [bc : ad : bd],$$

which is defined on the complement of the point $([1:0],[1:0])$. Let $\tilde{Y}$ be the Zariski closure of the graph of this rational map; it is easily checked to be the blowup of Y at the missing point.

The rational map $X \dashrightarrow Y$ induces a rational map $\tilde{X} \to \tilde{Y}$, which we claim extends to a true morphism; that is, the Zariski closure of its graph is again a graph. From this last point of view, it is enough to check at the level of closed points; the only issue is with the points in the exceptional fibers. So consider for instance a point in the fiber over $[0:1:0]$ (the other fiber behaves similarly). If we take the neighborhood $U = \operatorname{Spec}(k[x/y, z/y])$ of $[0:1:0]$ in X, its inverse image in $\tilde{X}$ is

$$\tilde{U} = \operatorname{Proj} k[x/y, z/y][e:f]/((x/y)f - (z/y)e).$$

This maps to Y taking $(x/y, z/y)[e:f]$ to $([1:z/y],[e:f])$. This means that we have defined a map $\tilde{U} \to \tilde{Y}$ except on the point $(1,0)[1:0]$, which maps to $([1:0],[1:0])$ in Y. To resolve this point, let's take the neighborhood $V = \operatorname{Spec}(k[b/a, d/c])$ of $([1:0],[1:0])$ in Y; its inverse image in $\tilde{Y}$ is

$$\tilde{V} = \operatorname{Proj} k[b/a, d/c][g:h]/((b/a)h - (d/c)g).$$

In a neighborhood of $(1,0)[1:0]$, in $\tilde{U}$, we can map to $\tilde{Y}$ taking $(x/y, z/y)[e:f]$ to $([1:z/y],[e:f])[f(z/y):e]$.

By similar considerations, we get a well-defined map $\tilde{X} \to \tilde{Y}$ near the other exceptional fiber, and hence a map defined everywhere. Again by similar considerations, we get a well-defined map $\tilde{Y} \to \tilde{X}$. The compositions of these maps must be the identity maps since this is true on an open dense subset, so we get the desired isomorphisms.

Sketch of alternate solution suggested by Cal Spicer: the closed subscheme of X consisting of the two reduced points $[1:0:0],[0:1:0]$ is cut out by the homogeneous ideal (xy, xz, yz, z^2), so we may view $\tilde{X}$ as a closed subscheme of $\mathbb{P}^2_k \times \mathbb{P}^3_k$. Similarly, if we identify $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ with the closed subscheme $V(sv - tu)$ in $\mathbb{P}^3_k \cong \operatorname{Proj} k[s,t,u,v]$ via the Segre embedding $([a:b],[c:d]) \mapsto [bd:ad:bc:ac]$, then the closed subscheme consisting of the reduced point $([1:0],[1:0]) \mapsto [0:0:0:1]$ is cut out by the homogeneous ideal (s,t,u), so we may view $\tilde{Y}$ as a closed subscheme of $\mathbb{P}^3_k \times \mathbb{P}^2_k$. One then checks that matching up factors induces the isomorphism $\tilde{X} \cong \tilde{Y}$.

5. (a) By the Jacobian criterion, the singular points are those points of X where the partial derivatives of $y^2z - x^3 - x^2z$ all vanish. They are

$$-3x^2 - 2xz, \qquad 2yz, \qquad y^2 - x^2.$$

From the partial in y, we see that either $y = 0$ or $z = 0$. If $y = 0$, then also $x = 0$ (from the partial in z) and we find the singular point $[0:0:1]$. If $z = 0$, then $x = 0$ (from the partial in x) and then $y = 0$ (from the partial in y) and we find no singular point.

Warning: if one checks in affine coordinates instead, one must make sure that one's singular points also lie on the curve!

(b) Let O be the point $[0:0:1]$. The map $[x:y:z] \mapsto [x:y]$ then defines a map $X - \{O\} \to \mathbb{P}^1_k$. To find the image, notice that the lowest-order terms of $y^2z - x^3 - x^2z$ at O are $y^2z - x^2z$, suggesting that the tangent lines are approaching $y = \pm x$ as we approach O. That in turn suggests that the map $X - \{O\} \to \mathbb{P}^1_k$ should induce an open immersion onto $\mathbb{P}^1_k - \{\pm 1\}$.

To check this, choose $[a:b] \in \mathbb{P}^1_k - \{\pm 1\}$. The formula $[x:y:z] \mapsto [x:y]$ defines a map $\mathbb{P}^2_k - \{O\} \to \mathbb{P}^1_k$, and the inverse image of $[a:b]$ is the line $y/x = b/a$, or $ay = bx$. To find the intersections of this line with X, we set $x = a, y = b, z = t$ and solve for t in the equation

$$b^2t = a^3 + a^2t$$

to get $t = a^3/(b^2 - a^2)$. The map $X - \{O\} \to \mathbb{P}^1_k - \{\pm 1\}$ is then the inverse of the map $[a:b] \mapsto [a:b:a^3/(b^2 - a^2)]$.

(c) It is enough to produce an exact sequence

$$0 \to k^* \to \mathrm{CaCl}(X) \to \mathbb{Z} \to 0$$

because any such sequence of abelian groups is obviously split. Since $X - \{O\} \cong \mathbb{P}^1_k \cong \{\pm 1\}$ and the latter obviously has trivial CaCl, we may replace X with $\operatorname{Spec} \mathcal{O}_{X,O}$ for this computation. But then the map to $\mathbb{Z}$ is the one measuring order of vanishing (or if you prefer, pull back along the surjection $\mathbb{P}^1_k \to X$ and measure order of vanishing at either $+1$ or -1, the answers being the same) and the kernel may be identified with k^* by again pulling back to $\mathbb{P}^1_k$ and taking the ratio of the values at $+1$ and -1.

6. We will first classify lines up to permutation of the variables. Suppose $[a : b : c : d]$ is a point of X contained in a line. Without loss of generality we may assume $d = 1$. Any line through $[a : b : c : 1]$ in $\mathbb{P}^3_{\mathbb{C}}$ then admits a parametric equation of the form $t \mapsto [a + et : b + ft : c + gt : 1]$ for some $e, f, g \in \mathbb{C}$ not all zero. We must have

$$(a + et)^3 + (b + ft)^3 + (c + gt)^3 + 1 = 0.$$

Without loss of generality we may assume $g \neq 0$ (by permuting variables), $g = 1$ (by rescaling e, f, g) and $c = 0$ (by shifting the point along the line). We then have

$$\begin{aligned} -1 &= a^3 + b^3 \\ 0 &= a^2 e + b^2 f \\ 0 &= a e^2 + b f^2 \\ -1 &= e^3 + f^3. \end{aligned}$$

From the first equation, we may assume without loss of generality that $a \neq 0$. Then $a^2 e = -b^2 f$ and $ae^2 = -bf^2$, so $(ae)^3 = (bf)^3$. If $e = 0$, then $f \neq 0$ from the fourth equation, so $b = 0$, and a and f are both cube roots of unity. This gives us a line of the form

$$t \mapsto [\zeta_3 : \zeta_3' t : t : 1]$$

for some cube roots of unity ζ_3, ζ_3'; in other words, this line is given by $a = \zeta_3 d, b = \zeta_3' c$. By permuting coordinates, this gives us a total of (3 choices for the pairing) $\times$ (3 choices for the ratio between each pair) $= 27$ lines.

If $e \neq 0$, then $f \neq 0$ from the fourth equation and $a/e = b/f$ (so also $b \neq 0$); also, $a^6 = b^6$, so $a = \zeta_6 b$ and $e = \zeta_6 f$ for some sixth root of unity ζ_6. But now on one hand, to have

$$-1 = a^3 + b^3 = a^3 + \zeta_6^3 b^3 = a^3 \pm a^3$$

we must have $\zeta_6^3 = 1$; on the other hand, to have

$$0 = a^2 e + b^2 f = \zeta_6^3 b^2 f + b^2 f = \pm b^2 f + b^2 f$$

we must have $\zeta_6^3 = -1$. So we get no more lines from this case.

7. Given a line bundle $\mathcal{L}$ on X, by definition there exists an open covering $\{U_i\}_{i \in I}$ of X such that $\mathcal{L}|_{U_i}$ admits a free generator s_i. Then the elements $s_i/s_j \in \mathcal{O}_X^*(U_i \cap U_j)$ form a 1-cocycle. In the other direction, given an open covering $\{U_i\}_{i \in I}$ of X and a 1-cocycle $t_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$, we may use the t_{ij} to define glueing data for the trivial line bundles on the U_i.