**Math 203C (Algebraic Geometry), UCSD, spring 2013**
**Problem Set 3 (due Wednesday, May 1)**

Solve the following problems, and turn in the solutions to *four* of them. Note: no classes April 22–26 because I'll be out of town.

1. Let $R$ be a ring.

   (a) Let $M$ be a finitely presented $R$-module. Prove that for any surjective $R$-module homomorphism $f : F \to M$ with $F$ finitely generated, $\ker(f)$ is also finitely generated.

   (b) Let $S$ be a finitely presented $R$-algebra. Prove that for any surjective $R$-algebra homomorphism $f : P \to M$ with $P$ finitely generated, $\ker(f)$ is also finitely generated.

   (c) Let $S$ be a finitely presented $R$-algebra which is finitely generated as an $R$-module. Prove that $S$ is also finitely presented as an $R$-module. Hint: the ring $R$ is the union of noetherian subrings; use the finite presentation hypothesis to reduce to working over such a subring.

2. Let $R \to S$ be a finite ring homomorphism. Prove that the following conditions are equivalent.

   (i) The morphism $R \to S$ is étale.

   (ii) The module $S$ is projective over both $R$ and $S \otimes_R S$ (via the multiplication map $S \otimes_R S \to S$).

   (iii) The $R$-module $S$ is projective and the map $S \to \mathrm{Hom}_R(S, R)$ taking $s$ to $t \mapsto \mathrm{Trace}(u \mapsto stu)$ is an isomorphism of $R$-modules. (We defined $\mathrm{Trace} : \mathrm{Hom}_R(S, S) \to R$ on a previous problem set.)

   Hint: for most of this you may assume $R$ is a local ring. Also, a finitely *presented* module is flat iff it is projective.

3. Let $R$ be a complete discrete valuation ring with fraction field $K$. Let $L$ be a finite separable extension of $K$. Let $S$ be the integral closure of $R$ in $L$; it is known (and you may assume without proof) that $S$ is a complete discrete valuation ring and $R \to S$ is a finite homomorphism. Prove that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is étale if and only if the following conditions hold.

   (i) The value groups of $R$ and $S$ coincide.

   (ii) The residue field of $S$ is separable over the residue field of $R$.

   For example, $\mathbb{Z}_p[\sqrt{p}]$ is not étale over $\mathbb{Z}_p$.

4. Let $k$ be an algebraically closed field. Put $X = \mathbb{P}^2_k$ and $Y = \mathbb{P}^1_k \times_{\mathrm{Spec}(k)} \mathbb{P}^1_k$.

(a) Prove that $X$ and $Y$ are not isomorphic as schemes over $k$. (Optional: omit "over $k$".)

(b) Let $Z$ be the blowup of $X$ at a point. Produce a one-parameter family of copies of $\mathbb{P}^1_k$ inside $Z$, any two of which are disjoint.

(c) Prove that there is a blowup of $X$ at two points which is isomorphic to a blowup of $Y$ at ~~two points~~ one point.

5. Let $k$ be a field of characteristic $\neq 2, 3$. Let $X$ be the curve $y^2z = x^3 + x^2z$ in $\mathbb{P}^2_k$.

   (a) Prove that $X$ is not smooth over $k$ by finding all of its singular points.

   (b) Show that $X$ minus its singular point(s) is isomorphic to an open subscheme of $\mathbb{P}^1_k$.

   (c) Suppose that $k$ is algebraically closed. Prove that the group of Cartier divisors on $X$ modulo principal Cartier divisors is isomorphic to $k^* \times \mathbb{Z}$.

6. Let $X$ be the threefold $x^3 + y^3 + z^3 + w^3 = 0$ in $\mathbb{P}^3_{\mathbb{C}}$. Prove that there are exactly 27 distinct lines of $\mathbb{P}^3_{\mathbb{C}}$ contained in $X$. We will see later that this is true for *every* smooth cubic threefold over an algebraically closed field of characteristic $\neq 3$.

7. Prove that for any scheme $X$, the group $\operatorname{Pic}(X)$ is isomorphic to the Čech cohomology group $\check{H}^1(X, \mathcal{O}_X^*)$. It can even be shown that agrees with the sheaf $H^1$ even though $\mathcal{O}_X^*$ is not a quasicoherent sheaf; see Hartshorne exercise III.4.4.