1. Suppose that $R$ is perfect and noetherian. For $I$ an ideal of $R$, put $I^{1/p} = \{x^{1/p} : x \in I\}$; this is again an ideal of $R$. We must then have $I = I^{1/p}$, as otherwise the sequence $I, I^{1/p}, I^{1/p^2}, \ldots$ would form an infinite ascending chain of ideals.

Suppose that $R$ is an integral domain. Then for any nonzero $x \in R$, the principal ideals $(x^p)$ and $(x)$ must coincide by the first paragraph, so $x = x^py$ for some $y \in R$. But then $1 = x^{p-1}y$, so $x$ is a unit in $R$; it follows that $R$ is a field.

In general, since $R$ is noetherian it has finitely many minimal prime ideals $I_1, \ldots, I_n$. By the first paragraph, $R/I_i$ is again perfect, and by the second paragraph it is a field. Since $R$ is perfect, it is reduced, so the map $R \to R/I_1 \oplus \cdots \oplus R/I_n$ is injective; it is also surjective by the Chinese remainder theorem.

2. Any very ample divisor is equivalent to a nonzero effective divisor (namely any hyperplane section), so its degree must be positive. Conversely, suppose $\deg(D) > 0$. To check that $D$ is ample, we must check that for any quasicoherent finitely generated sheaf $E$ on $X$, for $n$ large the sheaf $E \otimes \mathcal{O}(nD)$ is generated by global sections. If we fix a projective embedding and hence a choice of $\mathcal{O}(1)$, we can write $E$ as a quotient of a direct sum of various sheaves $\mathcal{O}(d_n)$, so it is enough to check the claim with $E$ equal to one of these. In particular, it is enough to check the case where $E = \mathcal{O}(k)$ for some divisor $E$. But by Riemann-Roch, for any closed points $P$ and $Q$, for $m_1, m_2 \in \{0, 1\}$, we have

$$h^0(E + nD - m_1P - m_2Q) = \deg(E) + n\deg(D) - m_1 - m_2 + 1 - g$$

for $n$ sufficiently large. For such $n$, $E + nD$ is very ample relative to $\text{Spec}(k)$.

3. (a) Since $\mathcal{L}_1$ is very ample relative to $\text{Spec}(k)$, it defines a closed immersion $j_1 : X \to \mathbb{P}^{m_1}$ for some $m_1 \geq 0$. Since $\mathcal{L}_2$ is generated by its global sections, it defines a morphism $j_2 : X \to \mathbb{P}^{m_2}$. Via the Segre embedding, $\mathcal{L}_1 \otimes \mathcal{L}_2$ defines a map $j_3 : X \to \mathbb{P}^{m_1+m_2}$ which must be a closed immersion because it factors as $j_1$ followed by a closed immersion. Hence $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample relative to $\text{Spec}(k)$.

(b) By a theorem from class, there exists $n_1$ such that for $n \geq n_1$, $\mathcal{L}_1^\otimes n_1$ is very ample relative to $\text{Spec}(k)$. By the definition of ampleness, there exists $n_2$ such that for $n \geq n_2$, $\mathcal{L}_1^\otimes n_2 \otimes \mathcal{L}_2$ is generated by global sections. By (a), the claim now follows for $n \geq n_1 + n_2$.

(c) Note that $S$ is invariant under positive scalar multiplication (by the definition of ampleness). Also, we proved in class that the tensor product of two ample line bundles is ample, which proves that $S$ is convex. To prove that $S$ is ample, note that for any point $s \in S$ and any $t \in \mathbb{Q}^n$, by (b), any point on the segment from $s$ to $t$ sufficiently close to $s$ is in $S$. If we take $t$ to run over the vertices of a simplex containing $s$ in its interior, we produce another simplex containing $s$ in
4. Let $X_1, X_2, X_3, X_4$ be the blowups of $X$ at the four indicated ideals. We claim that the only isomorphism among these is $X_1 \cong X_3$. We check that $X_1 \not\cong X_2$ by noticing that the inverse image ideal sheaf of $(x, y^2)$ on $X_1$ is not locally principal: on the coordinate chart $\text{Spec } k[y, x/y]$ it has the form $(y(x/y), y^2)$.

We check that $X_1 \cong X_3$ by noticing first that the inverse image ideal sheaf of $(x_2, xy, y^2)$ on $X_1$ is locally principal: on the chart $\text{Spec } k[y, x/y]$ of $X_1$ it is generated by $y^2$, while on the chart $\text{Spec } k[x, y/x]$ they are both generated by $x^2$. This yields the map $X_3 \to X_1$. In the other direction, write

$$X_3 = \text{Proj } k[x, y][a, b, c]/(ay - bx, by - cx, ac - b^2).$$

On the coordinate chart

$$X_{3,a} \cong k[x, y, b/a, c/a]/(y - (b/a)x, (b/a)y - (c/a)x, c/a - (b/a)^2) \cong k[x, b/a],$$

the inverse image ideal sheaf of $(x, y)$ is generated by $x$, and similarly on $X_{3,c}$ it is generated by $y$. On

$$X_{3,b} \cong k[x, y, a/b, c/b]/((a/b)y - x, y - (c/b)x, (a/b)(c/b) - 1),$$

the inverse image is generated by each of $x$ and $y$ (which differ by a unit). This yields a map $X_1 \to X_3$, so $X_1 \cong X_3$.

We check that $X_1 \not\cong X_4$ and $X_2 \not\cong X_4$ by writing

$$X_4 = \text{Proj } k[x, y][a, b]/(ay^2 - bx^2).$$

On the coordinate chart

$$X_{4,b} \cong k[x, y, a/b]/((a/b)y^2 - x^2),$$

neither the inverse image ideal sheaf of $(x, y)$ nor $(x, y^2)$ is locally principal at $x = y = a/b = 0$.

5. We start by blowing up at $(0, 0)$ to get $X_1$. Let $Z_1$ be the inverse image of $Z$ in $X_1$; it consists of the exceptional divisor $E_1$ plus the strict transform $Z_1$. In the chart $\text{Spec } k[y, x/y]$, $E_1$ is cut out by $y$ and $Z_1$ by $1 - y^2(x/y)^4 - y^3(x/y)^4$, and they do not meet. We thus need only consider the other chart $\text{Spec } k[x, y/x]$, where $E_1$ is cut out by $x$ and $Z_1$ by $(y/x)^2 - x^2 - x^3$. The only intersection is at $x = y/x = 0$, so we blow up there next. We write $E_1$ (again) for the strict transform of $E_1$, $E_2$ for the new exceptional divisor, and $Z_2$ for the strict transform of $Z_1$. In the chart $\text{Spec } k[x, y/x^2]$, $E_1$ is cut out by $y/x^2$, $E_2$ by $x$, and $Z_2$ by $(y/x^2)^2 - 1 - x$. These are all smooth and meet transversely. In the other chart $\text{Spec } k[y/x, x^2/y]$, $E_1$ is cut out by $x^2/y$, $E_2$ by $y/x$, and $Z_2$ by $1 - (x^2/y)^2 - (y/x)(x^2/y)^3$, and again all of these are smooth and meet transversely.
6. (a) Since the claim is local on $X$, we may assume $X = \text{Spec } R$ with $R$ noetherian. Let $I_Y = (f_1, \ldots, f_m)$ and $I_Z = (g_1, \ldots, g_n)$ be the ideals cutting out $Y$ and $Z$, so that $Y \times_X Z$ is cut out by $I_Y + I_Z = (f_1, \ldots, f_m, g_1, \ldots, g_n)$. We may then view $\tilde{X}$ as a closed subscheme of $P = \text{Proj } R[x_1, \ldots, x_m, y_1, \ldots, y_n]$. The proper transforms of $Y$ and $Z$ are the intersections of this subscheme with the closed subschemes $P_1 = \text{Proj } R[x_1, \ldots, x_m]$ and $P_2 = \text{Proj } R[y_1, \ldots, y_n]$ of $P$ (via the graded maps sending one of the two sets of generators to 0); but $P_1$ and $P_2$ are themselves disjoint.

(b) Take $X = \mathbb{A}^2_k$, $Y = V(y)$, and $Z = V(y - x^2)$. Then the reduced closed subscheme underlying $Y \times_X Z$ is the closed point $(0, 0)$, so we get the standard blowup. In the coordinate chart $\text{Spec } k[x, y/x]$, the strict transforms of $Y$ and $Z$ are cut out by $x$ and $y/x - x$, which meet at $x = y/x = 0$. 