Math 203C (Algebraic Geometry), UCSD, spring 2013
Solutions for problem set 4

1. Suppose that R is perfect and noetherian. For I an ideal of R, put $I^{1/p} = \{x^{1/p} : x \in I\}$; this is again an ideal of R. We must then have $I = I^{1/p}$, as otherwise the sequence $I, I^{1/p}, I^{1/p^2}, \dots$ would form an infinite ascending chain of ideals.

   Suppose that R is an integral domain. Then for any nonzero $x \in R$, the principal ideals (x^p) and (x) must coincide by the first paragraph, so $x = x^p y$ for some $y \in R$. But then $1 = x^{p-1}y$, so x is a unit in R; it follows that R is a field.

   In general, since R is noetherian it has finitely many minimal prime ideals $I_1, \dots, I_n$. By the first paragraph, R/I_i is again perfect, and by the second paragraph it is a field. Since R is perfect, it is reduced, so the map $R \to R/I_1 \oplus \cdots \oplus R/I_n$ is injective; it is also surjective by the Chinese remainder theorem.

2. Any very ample divisor is equivalent to a nonzero effective divisor (namely any hyperplane section), so its degree must be positive. Conversely, suppose $\deg(D) > 0$. To check that D is ample, we must check that for any quasicoherent finitely generated sheaf $\mathcal{E}$ on X, for n large the sheaf $\mathcal{E} \otimes \mathcal{O}(nD)$ is generated by global sections. If we fix a projective embedding and hence a choice of $\mathcal{O}(1)$, we can write $\mathcal{E}$ as a quotient of a direct sum of various sheaves $\mathcal{O}(d_n)$, so it is enough to check the claim with $\mathcal{E}$ equal to one of these. In particular, it is enough to check the case where $\mathcal{E} = \mathcal{O}(E)$ for some divisor E. But by Riemann-Roch, for any closed points P and Q, for $m_1, m_2 \in \{0, 1\}$, we have

   $$h^0(E + nD - m_1P - m_2Q) = \deg(E) + n\deg(D) - m_1 - m_2 + 1 - g$$

   for n sufficiently large. For such n, $E + nD$ is very ample relative to $\mathrm{Spec}(k)$.

3. (a) Since $\mathcal{L}_1$ is very ample relative to $\mathrm{Spec}(k)$, it defines a closed immersion $j_1 : X \to \mathbb{P}^{n_1}_k$ for some $n_1 \geq 0$. Since $\mathcal{L}_2$ is generated by its global sections, it defines a morphism $j_2 : X \to \mathbb{P}^{n_2}_k$. Via the Segre embedding, $\mathcal{L}_1 \otimes \mathcal{L}_2$ defines a map $j_3 : X \to \mathbb{P}^{n_3}_k$ which must be a closed immersion because it factors as j_1 followed by a closed immersion. Hence $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample relative to $\mathrm{Spec}(k)$.

   (b) By a theorem from class, there exists n_1 such that for $n \geq n_1$, $\mathcal{L}_1^{\otimes n_1}$ is very ample relative to $\mathrm{Spec}(k)$. By the definition of ampleness, there exists n_2 such that for $n \geq n_2$, $\mathcal{L}_1^{\otimes n_2} \otimes \mathcal{L}_2$ is generated by global sections. By (a), the claim now follows for $n \geq n_1 + n_2$.

   (c) Note that S is invariant under positive scalar multiplication (by the definition of ampleness). Also, we proved in class that the tensor product of two ample line bundles is ample, which proves that S is convex. To prove that S is ample, note that for any point $s \in S$ and any $t \in \mathbb{Q}^n$, by (b), any point on the segment from s to t sufficiently close to s is in S. If we take t to run over the vertices of a simplex containing s in its interior, we produce another simplex containing s in its interior consisting of points of S; by convexity, S contains a neighborhood of s. Hence S is open.

4. Let X_1, X_2, X_3, X_4 be the blowups of X at the four indicated ideals. We claim that the only isomorphism among these is $X_1 \cong X_3$. We check that $X_1 \not\cong X_2$ by noticing that the inverse image ideal sheaf of (x, y^2) on X_1 is not locally principal: on the coordinate chart $\operatorname{Spec} k[y, x/y]$ it has the form $(y(x/y), y^2)$.

   We check that $X_1 \cong X_3$ by noticing first that the inverse image ideal sheaf of (x_2, xy, y^2) on X_1 is locally principal: on the chart $\operatorname{Spec} k[y, x/y]$ of X_1 it is generated by y^2, while on the chart $\operatorname{Spec} k[x, y/x]$ they are both generated by x^2. This yields the map $X_3 \to X_1$. In the other direction, write

   $$X_3 = \operatorname{Proj} k[x, y][a, b, c]/(ay - bx, by - cx, ac - b^2).$$

   On the coordinate chart

   $$X_{3,a} \cong k[x, y, b/a, c/a]/(y - (b/a)x, (b/a)y - (c/a)x, c/a - (b/a)^2) \cong k[x, b/a],$$

   the inverse image ideal sheaf of (x, y) is generated by x, and similarly on $X_{3,c}$ it is generated by y. On

   $$X_{3,b} \cong k[x, y, a/b, c/b]/((a/b)y - x, y - (c/b)x, (a/b)(c/b) - 1),$$

   the inverse image is generated by each of x and y (which differ by a unit). This yields a map $X_1 \to X_3$, so $X_1 \cong X_3$.

   We check that $X_1 \not\cong X_4$ and $X_2 \not\cong X_4$ by writing

   $$X_4 = \operatorname{Proj} k[x, y][a, b]/(ay^2 - bx^2).$$

   On the coordinate chart

   $$X_{4,b} \cong k[x, y, a/b]/((a/b)y^2 - x^2),$$

   neither the inverse image ideal sheaf of (x, y) nor (x, y^2) is locally principal at $x = y = a/b = 0$.

5. We start by blowing up at $(0, 0)$ to get X_1. Let Z_1 be the inverse image of Z in X_1; it consists of the exceptional divisor E_1 plus the strict transform Z_1. In the chart $\operatorname{Spec} k[y, x/y]$, E_1 is cut out by y and Z_1 by $1 - y^2(x/y)^4 - y^3(x/y)^4$, and they do not meet. We thus need only consider the other chart $\operatorname{Spec} k[x, y/x]$, where E_1 is cut out by x and Z_1 by $(y/x)^2 - x^2 - x^3$. The only intersection is at $x = y/x = 0$, so we blow up there next. We write E_1 (again) for the strict transform of E_1, E_2 for the new exceptional divisor, and Z_2 for the strict transform of Z_1. In the chart $\operatorname{Spec} k[x, y/x^2]$, E_1 is cut out by y/x^2, E_2 by x, and Z_2 by $(y/x^2)^2 - 1 - x$. These are all smooth and meet transversely. In the other chart $\operatorname{Spec} k[y/x, x^2/y]$, E_1 is cut out by x^2/y, E_2 by y/x, and Z_2 by $1 - (x^2/y)^2 - (y/x)(x^2/y)^3$, and again all of these are smooth and meet transversely.

6. (a) Since the claim is local on X, we may assume $X = \operatorname{Spec} R$ with R noetherian. Let $I_Y = (f_1, \dots, f_m)$ and $I_Z = (g_1, \dots, g_n)$ be the ideals cutting out Y and Z, so that $Y \times_X Z$ is cut out by $I_Y + I_Z = (f_1, \dots, f_m, g_1, \dots, g_n)$. We may then view $\tilde{X}$ as a closed subscheme of $P = \operatorname{Proj} R[x_1, \dots, x_m, y_1, \dots, y_n]$. The proper transforms of Y and Z are the intersections of this subscheme with the closed subschemes $P_1 = \operatorname{Proj} R[x_1, \dots, x_m]$ and $P_2 = \operatorname{Proj} R[y_1, \dots, y_n]$ of P (via the graded maps sending one of the two sets of generators to 0); but P_1 and P_2 are themselves disjoint.

   (b) Take $X = \mathbb{A}^2_k$, $Y = V(y)$, and $Z = V(y - x^2)$. Then the reduced closed subscheme underlying $Y \times_X Z$ is the closed point $(0,0)$, so we get the standard blowup. In the coordinate chart $\operatorname{Spec} k[x, y/x]$, the strict transforms of Y and Z are cut out by x and $y/x - x$, which meet at $x = y/x = 0$.