Math 203C (Algebraic Geometry), UCSD, spring 2013
Problem Set 4 (due Wednesday, May 8)

Solve the following problems, and turn in the solutions to *four* of them. (Note that there is a second page!)

1. In case you need any convincing that nonnoetherian rings are important in algebraic geometry, here is an example. Let R be a ring of characteristic p which is perfect, i.e., the p-power Frobenius endomorphism $x \mapsto x^p$ is bijective. Prove that R cannot be noetherian unless it is a direct sum of finitely many fields.

2. Let C be a smooth projective connected curve over an algebraically closed field k. Prove that a divisor D on C is ample if and only if $\deg(D) > 0$. Hint: use Riemann-Roch.

3. Let X be a projective scheme over a field k.

   (a) Let $\mathcal{L}_1, \mathcal{L}_2$ be line bundles on X such that $\mathcal{L}_1$ is very ample relative to $\mathrm{Spec}(k)$ and $\mathcal{L}_2$ is generated by global sections. Prove that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample relative to $\mathrm{Spec}(k)$.

   (b) Let $\mathcal{L}_1, \mathcal{L}_2$ be line bundles on X such that $\mathcal{L}_1$ is ample. Prove that for any sufficiently large positive integer n, $\mathcal{L}_1^{\otimes n} \otimes \mathcal{L}_2$ is very ample relative to $\mathrm{Spec}(k)$, and hence ample.

   (c) Let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be line bundles on X. Let S be the subset of $\mathbb{Q}^n$ consisting of tuples $(r_1, \dots, r_n)$ such that for some positive integer d, $r_1 d, \dots, r_n d \in \mathbb{Z}$ and $\mathcal{L}_1^{\otimes r_1 d} \otimes \cdots \otimes \mathcal{L}_n^{\otimes r_n d}$ is ample. Prove that S is an open convex subset of $\mathbb{Q}^n$. This property is sometimes summarized by saying that *the ample cone is open.*

4. Let k be a field. Form the blowups of $\mathbb{A}^2_k = \mathrm{Spec}\, k[x, y]$ at the closed subschemes defined by the ideals

   $$(x, y), \qquad (x, y^2), \qquad (x^2, xy, y^2), \qquad (x^2, y^2).$$

   Which of these blowups are isomorphic to each other? Hint: use the universal property of blowups, but don't forget to produce maps in both directions in cases where you are claiming isomorphisms.

5. Let k be a field of characteristic 0. Let Z be the closed subscheme of $\mathbb{A}^2_k$ cut out by $y^2 - x^4 - x^5$. Construct a sequence of point blowups starting from $\mathbb{A}^2_k$ under which the inverse image of Z consists of smooth components meeting transversely (i.e., two at a time with distinct tangent directions). For the record, this singularity is an example of a *tacnode.*

6. (a) Let X be a locally noetherian scheme. Let Y and Z be two irreducible closed subschemes of X, neither one containing the other. Let $\tilde{X}$ be the blowup of X in the closed subscheme $Y \times_X Z$. Prove that the strict transforms of Y and Z in $\tilde{X}$ are disjoint from each other. Hint: work locally and choose generators.

(b) Give an explicit example where (a) fails if we instead blow up in the reduced closed subscheme underlying $Y \times_X Z$ (i.e., the set-theoretic intersection).