Math 203C (Algebraic Geometry), UCSD, spring 2013
Solutions for problem set 5

1. (a) We have $\omega_{X/k} = \omega_{\mathbb{P}^3_k/k} \otimes \mathcal{O}(4) = \mathcal{O}$ because the canonical sheaf on $\mathbb{P}^n_k$ is $\mathcal{O}(-n-1)$. To compute $H^1(X, \mathcal{O}_X)$, we may write it as $H^1(\mathbb{P}^3_k, f_*\mathcal{O}_X)$ for $f : X \to \mathbb{P}^3_k$ the closed immersion. Using the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3_k}(-4) \to \mathcal{O}_{\mathbb{P}^3_k} \to f_*\mathcal{O}_X \to 0$$

we have an exact sequence

$$H^1(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}) \to H^1(X, \mathcal{O}_X) \to H^2(\mathbb{P}^3_K, \mathcal{O}_{\mathbb{P}^3_k}(-4)).$$

The outer terms are both 0, so the inside term is too.

(b) In this case, we have $\omega_{X/k} = \omega_{\mathbb{P}^4_k/k} \otimes \mathcal{O}(2+3) = \mathcal{O}$. To compute $H^1(X, \mathcal{O}_X)$, let Q be the degree 2 hypersurface Let $f : X \to Q$ and $g : Q \to \mathbb{P}^4_k$ be the closed immersions. Using the exact sequence

$$0 \to \mathcal{O}_Q(3) \to \mathcal{O}_Q \to f_*\mathcal{O}_X \to 0$$

we have an exact sequence

$$H^1(Q, \mathcal{O}_Q) \to H^1(X, \mathcal{O}_X) \to H^2(Q, \mathcal{O}_Q(3)).$$

As in (a), we compute that the outside terms are 0.

2. (a) For $j : X \to \mathbb{P}^3_k$ the closed immersion, we have $\omega_X = j^*(\omega_{\mathbb{P}^3_k} \otimes \mathcal{O}(d)) = j^*(\mathcal{O}(-d-4))$. Let H_1, H_2 be two hypersurfaces of degree d in in $\mathbb{P}^3_k$; we then have $K \cdot K = (X \cap H_1) \cdot (X \cap H_2)$. For generic choices of H_1, H_2, the intersection $X \cap H_1 \cap H_2$ is transverse and consists of $d(d-4)^2$ points, proving the claim.

(b) Write $X = C_1 \times_k C_2$. Let K_1, K_2 be the canonical divisors on C_1, C_2. If we identify K_1 with the divisor $K_1 \times C_2$ and similarly for K_2, we then have $K = K_1 + K_2$ and (since $K_1 \cdot K_1 = K_2 \cdot K_2 = 0$)

$$K \cdot K = 2K_1 \cdot K_2 = 2\deg(K_1)\deg(K_2) = 2(2g_1 - 2)(2g_2 - 2) = 8(g_1 - 1)(g_2 - 1).$$

3. Let $\mathcal{I}$ be the ideal sheaf defining D. Then ω_C is isomorphic to the dual of $\mathcal{I}/\mathcal{I}^2 = \Delta^*(\mathcal{I})$. We thus have

$$D \cdot D = \deg_D(\Delta^*(\mathcal{I})^\vee) = -\deg_C(\Omega_{C/k}) = 2 - 2g.$$

4. (a) Put

$$E = D - \frac{D \cdot H}{H \cdot H} H,$$

so that $E \cdot H = 0$ and so $E \cdot E = E \cdot D$. By the Hodge index theorem,

$$0 \geq E \cdot E = E \cdot D = D \cdot D - \frac{D \cdot H}{H \cdot H}(D \cdot H).$$

This yields the claimed inequality.

(b) Put $H = C + C'$, which is ample. Since $C \cdot C = 0$, $C \cdot C' = 1$, $C' \cdot C' = 0$, we have $H \cdot H = 2$. Also, if we put $J = C - C'$, then $J \cdot J = -2$ and $H \cdot J = 0$; in particular, H and J are orthogonal.

Put

$$E = D - \frac{D \cdot H}{2} H + \frac{D \cdot J}{2} J,$$

so that $E \cdot H = E \cdot J = 0$, and so $E \cdot E = E \cdot D$. By the Hodge index theorem,

$$\begin{aligned} 0 &\geq E \cdot E \\ &= E \cdot D - \frac{1}{2}(D \cdot H)^2 + \frac{1}{2}(D \cdot J)^2 \\ &= E \cdot D - \frac{1}{2}(D \cdot C + D \cdot C')^2 + \frac{1}{2}(D \cdot C - D \cdot C')^2 \\ &= E \cdot D - 2(D \cdot C)(D \cdot C'). \end{aligned}$$

This yields the claimed inequality.

5. (a) Let x be an intersection point of D and F; then in particular x is an $\mathbb{F}_q$-rational point of X. Let t be a uniformizer for C at x. Let t_1, t_2 be the pullbacks of t to the two copies of C in X; then D is locally (at x) cut out by $t_2 - t_1$ while F is locally cut out by $t_2 - t_1^q$. The differentials of these functions are $dt_2 - dt_1$ and dt_2, which are linearly independent. This gives the desired transversality.

   (b) Let $\mathcal{I}$ be the ideal sheaf defining D. Then the restriction of the ideal sheaf defining F, when pulled back to C, is isomorphic to $\varphi^*(\mathcal{I}/\mathcal{I}^2)^\vee$, so its degree is $q(2 - 2g)$.

6. Put $N = \#C(\mathbb{F}_q)$. We compute $C \cdot D = C' \cdot D = 1$, $C \cdot F = q$, $C' \cdot F = 1$; from a previous problem we have $D \cdot D = 2 - 2g$. Put $E = rD + sF$ for some $r, s \in \mathbb{Z}$. By a previous problem,

$$E \cdot E \leq 2(E \cdot C)(E \cdot C').$$

The left side of this inequality equals

$$(rD + sF) \cdot (rD + sF) = r^2(2 - 2g) + 2rsN + s^2 q(2 - 2g).$$

By a previous problem, the right side equals

$$2(rD \cdot C + sF \cdot C)(rD \cdot C' + sF \cdot C') = 2(r + qs)(r + s).$$

Therefore we have

$$r^2(2 - 2g) + 2rsN + s^2 q(2 - 2g) \leq 2(r + qs)(r + s)$$

or equivalently

$$2gr^2 + 2(q + 1 - N)rs + 2gqs^2 \geq 0.$$

But this is true for any r, s, so the quadratic form on the left side must be positive definite; in particular, its discriminant

$$4(q + 1 - N)^2 - 4(2gq)(2g)$$

must be nonpositive. This gives the desired inequality.