Math 203C (Algebraic Geometry), UCSD, spring 2013
Problem Set 5 (due Friday, May 17)

Solve the following problems, and turn in the solutions to four of them. Throughout this problem set, let $k$ be an algebraically closed field (of arbitrary characteristic unless otherwise specified).

1. A $K3$ surface over $k$ is a surface $X$ for which $\omega_{X/k} \cong \mathcal{O}_X$ (the canonical divisor is trivial) and $H^1(X, \mathcal{O}_X) = 0$. Prove that the following give examples of $K3$ surfaces.
   
   (a) Any smooth surface of degree 4 in $\mathbb{P}^3_k$.
   
   (b) A smooth complete intersection of a degree 2 and a degree 3 hypersurface in $\mathbb{P}^4_k$.

2. (a) Let $X$ be a smooth surface of degree $d$ in $\mathbb{P}^3_k$. Prove that $K \cdot K = d(d - 4)$. 
   
   (b) Let $X$ be the product of two curves of genera $g_1, g_2$. Prove that $K \cdot K = 8(g_1 - 1)(g_2 - 1)$.

3. Let $C$ be a curve of genus $g$ over $k$, take $X = C \times_k C$, and let $D$ be the image of the diagonal $\Delta : C \to X$. Prove that $D \cdot D = 2 - 2g$.

4. (a) Let $H$ be an ample divisor on $X$. Prove that for any divisor $D$ on $X$,
   
   $$(D \cdot D)(H \cdot H) \leq (D \cdot H)^2.$$ 
   
   Hint: orthogonalize.
   
   (b) Take $X = C \times_k C'$ for $C, C'$ two curves over $k$. Prove that for any divisor $D$ on $X$,
   
   $$D \cdot D \leq 2(D \cdot C)(D \cdot C')$$
   
   where $C$ is identified with the divisor $C \times \{x'\}$ for some (any) closed point $x' \in C'$, and similarly for $C'$. Hint: orthogonalize again, this time using $C + C'$ and $C - C'$.

5. In this problem and the next, we reconstruct one of Weil’s proofs of the Riemann hypothesis for curves over a finite field using the Hodge index theorem. Take $k$ to be an algebraic closure of a finite field $\mathbb{F}_q$. Let $C$ be a curve of genus $g$ over a finite field $\mathbb{F}_q$ and write $C_k$ for $C \times \text{Spec}(\mathbb{F}_q) \text{Spec}(k)$. Put $X = C \times_k C$, let $D$ be the diagonal in $X$, and let $F$ be the graph of the $q$-power Frobenius map $\varphi : C \to C$.
   
   (a) Prove that $D$ and $F$ meet transversally, so $D \cdot F = \#C(\mathbb{F}_q)$. Hint: work locally around an intersection point.
   
   (b) Prove that $F \cdot F = q(2 - 2g)$.

6. With notation as in the previous problem, prove that

   $$|\#C(\mathbb{F}_q) - 1 - q| \leq 2g\sqrt{q}.$$ 
   
   Hint: consider $rD + sF$ for varying $r, s$. 

1