Math 203C (Algebraic Geometry), UCSD, spring 2013
Problem Set 5 (due *Friday*, May 17)

Solve the following problems, and turn in the solutions to *four* of them. Throughout this problem set, let k be an algebraically closed field (of arbitrary characteristic unless otherwise specified).

1. A *K3 surface* over k is a surface X for which $\omega_{X/k} \cong \mathcal{O}_X$ (the canonical divisor is trivial) and $H^1(X, \mathcal{O}_X) = 0$. Prove that the following give examples of K3 surfaces.
   (a) Any smooth surface of degree 4 in $\mathbb{P}^3_k$.
   (b) A smooth complete intersection of a degree 2 and a degree 3 hypersurface in $\mathbb{P}^4_k$.

2. (a) Let X be a smooth surface of degree d in $\mathbb{P}^3_k$. Prove that $K \cdot K = d(d-4)^2$.
   (b) Let X be the product of two curves of genera g_1, g_2. Prove that $K \cdot K = 8(g_1 - 1)(g_2 - 1)$.

3. Let C be a curve of genus g over k, take $X = C \times_k C$, and let D be the image of the diagonal $\Delta : C \to X$. Prove that $D \cdot D = 2 - 2g$.

4. (a) Let H be an ample divisor on X. Prove that for any divisor D on X,

$$(D \cdot D)(H \cdot H) \leq (D \cdot H)^2.$$

   Hint: orthogonalize.
   (b) Take $X = C \times_k C'$ for C, C' two curves over k. Prove that for any divisor D on X,

$$D \cdot D \leq 2(D \cdot C)(D \cdot C')$$

   where C is identified with the divisor $C \times \{x'\}$ for some (any) closed point $x' \in C'$, and similarly for C'. Hint: orthogonalize again, this time using $C + C'$ and $C - C'$.

5. In this problem and the next, we reconstruct one of Weil's proofs of the Riemann hypothesis for curves over a finite field using the Hodge index theorem. Take k to be an algebraic closure of a finite field $\mathbb{F}_q$. Let C be a curve of genus g over a finite field $\mathbb{F}_q$ and write C_k for $C \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(k)$. Put $X = C \times_k C$, let D be the diagonal in X, and let F be the graph of the q-power Frobenius map $\varphi : C \to C$.
   (a) Prove that D and F meet transversally, so $D \cdot F = \#C(\mathbb{F}_q)$. Hint: work locally around an intersection point.
   (b) Prove that $F \cdot F = q(2 - 2g)$.

6. With notation as in the previous problem, prove that

$$|\#C(\mathbb{F}_q) - 1 - q| \leq 2g\sqrt{q}.$$

   Hint: consider $rD + sF$ for varying r, s.