Math 203C (Algebraic Geometry), UCSD, spring 2013
Solutions for problem set 6

1. We may assume that $U = \mathrm{Spec}(A)$ is affine. Since f is affine, $U' = f^{-1}(U)$ is also affine; write it as $\mathrm{Spec}(B)$. Choose elements $s_1, \dots, s_m$ of B which form a basis of $\mathrm{Frac}(B)$ over $\mathrm{Frac}(A)$. For $n > 0$ sufficiently large, the sections $s_i g^{\otimes n}$ of $\mathcal{L}^{\otimes n}$ over Y_g extend to sections b_i of $\mathcal{L}^{\otimes n}$ over Y (namely, cover Y with finitely many open affines, realize that we know this over each one, then choose a uniform value of n). View the b_i as sections of $f_*(\mathcal{L}^{\otimes n})$ and then use to define a homomorphism u; this has the desired effect at x, and hence also in a neighborhood because the kernel and cokernel of u are coherent sheaves, so their supports are closed subsets of X.

2. We induct on the dimension of X. Set notation as in the previous exercise. Since $f^*\mathcal{L}$ is ample, there exist g, U of the prescribed form, so we may construct a morphism u as claimed for sufficiently large n. Then over some neighborhood V of x, the adjunction map $\mathcal{L} \to f_* f^* \mathcal{L}$ is not only injective but split.

   We first check that $\mathcal{L}^{\otimes n}$ is generated by global sections for some $n > 0$ (and hence for all multiples of n, i.e., for all sufficiently divisible n). Namely, choose some n for which $f^*(\mathcal{L}^{\otimes n})$ is generated by global sections and there exists u as in the previous problem; then the stalk of $\mathcal{L}^{\otimes n}$ at each point of V is generated by global sections. By the induction hypothesis, the same is true at each point of $X - V$ if we replace n by a suitable multiple, so the claim holds.

   We next observe that by a similar argument, for any given coherent sheaf $\mathcal{F}$ on X, there exist a positive integer n and an open subset V of X on which $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. But using the previous paragraph, we see that same is true (for the same V) for *every* sufficiently large n. We may thus again apply the induction hypothesis to $X - V$ to conclude.

3. Every line of $\mathbb{P}^4_k$ contained in X must be contained in a hyperplane section, which is a cubic passing through $P_1, \dots, P_5$. This cubic must generate as the union of a line L plus a possibly degenerate conic C. The intersection of this cubic with a generic cubic through $P_1, \dots, P_5$ consists of $P_1, \dots, P_5$ plus 4 more points. For L to embed into $\mathbb{P}^4_k$ with degree 1, exactly 1 of the 4 extra intersections must lie on L, which means that the other 2 intersections of the cubic with L must be among the 5 given points. One way this could happen is for L to be one of the 5 exceptional divisors. The other way is for L to be a line through two of $P_1, \dots, P_5$; that produces $\binom{5}{2} = 10$ lines.

   The other possibility is for C to be nondegenerate and to embed into $\mathbb{P}^4_k$ with degree 1. This only happens if C is the unique conic through all 5 points, so we end up with $5 + 10 + 1 = 16$ lines.

4. Let C_0 and C_∞ be the zero and pole loci of f. It was proved in class that we can choose π so that the inverse image of $C_0 \cup C_\infty$ is a divisor of strict normal crossings, so we may as well assume that we start from this situation.

We win if we can ensure that C_0 and C_∞ are disjoint. Suppose to the contrary that they cross at a point P, necessarily transversely. Let m and n be the multiplicities of C_0 and C_∞ at P. Now blow up at P and let E be the exceptional divisor. If $m > n$, then E belongs to the zero divisor with multiplicity $m - n$; if $m < n$, then E belongs to the pole divisor with multiplicity $n - m$; if $m = n$, then E belongs to neither.

Let us induct now primarily on the maximum value of $m + n$, and secondarily on the number of occurrences of this maximum value. Then the previous operation always decreases our count, so we must terminate after finitely many steps in the desired situation.

5. Let H be a very ample divisor on Y; then H is linearly equivalent to an effective divisor H' containing P and to an effective divisor H'' not containing P. Put $C' = \pi^*(H')$, $C'' = \pi^*(H'')$, and let D be the proper transform of H'. Then $C' = D + mC$ for some positive integer m. But C' and C'' are linearly equivalently, and obviously $C'' \cdot C = 0$, so $0 = C' \cdot C = C\dot{D} + mC^2$. Since $C \cdot D > 0$, we then have $C^2 < 0$.

6. Let P, Q be two points in $\mathbb{P}^2_k$. Let X be the blowup at these two points and let E_P, E_Q be the exceptional divisors. Let L be the line PQ and let $\tilde{L}$ be its proper transform; then $\tilde{L}^2 = L^2 - 2 = -1$, so $\tilde{L}$ is a (-1)-curve. We may thus blow it down to get a new smooth projective surface Y.

   The new surface Y contains two distinct families of pairwise disjoint lines. One of these families is obtained by taking lines through P in $\mathbb{P}^2_k$, taking their proper transforms, then mapping these down to Y. The other family is obtained similarly using Q.

   Each of these families is parametrized by $\mathbb{P}^1_k$, so we get a projective morphism $Y \to \mathbb{P}^1_k \times_k \mathbb{P}^1_k$. This is a bijection between smooth projective varieties, so it must be an isomorphism.