Solve the following problems, and turn in the solutions to four of them. No homework due Wednesday, May 22 due to qualifying exams.

1. Let $f : Y \to X$ be a finite surjective morphism between integral noetherian schemes. Let $x, y$ be the generic points of $X, Y$. Let $L$ be a line bundle on $Y$, let $U$ be a neighborhood of $x$, let $g$ be a global section of $L$, and suppose that $y \in Y_g \subseteq f^{-1}(U)$. Prove that for any sufficiently large $n > 0$, there exists a homomorphism $u : O_X \to f^*(L^\otimes n)$ of $O_X$-modules for some $m > 0$ which is an isomorphism over some neighborhood of $x$.

2. Let $f : Y \to X$ be a finite surjective morphism of proper integral schemes over a field $k$. Let $L$ be a line bundle on $X$. Prove that if $f^* L$ is ample, then so is $L$. Hint: use the previous exercise to reduce from $X$ to a closed subscheme of lower dimension.

3. Let $k$ be an algebraically closed field. Let $X$ be the blowup of $\mathbb{P}_k^2$ at five distinct closed points $P_1, \ldots, P_5$, no three of which are collinear, viewed as a degree 4 Del Pezzo surface in $\mathbb{P}_k^4$ (via the linear system $|3L - P_1 - \cdots - P_5|$). Prove that $X$ contains exactly 16 lines of $\mathbb{P}_k^4$.

4. Let $f$ be a rational function on a smooth projective connected surface $X$ over an algebraically closed field $k$. Prove that there exists a morphism $\pi : \widetilde{X} \to X$ which is a composition of monoidal transformations such that $\pi^*(f)$ defines a morphism $\widetilde{X} \to \mathbb{P}_k^1$. Hint: the key point is to separate the zero locus and the pole locus.

5. Let $C$ be an irreducible curve on a smooth projective connected surface $X$ over an algebraically closed field $k$. Suppose that there exists a morphism $\pi : X \to Y$ to a projective (but not necessarily smooth) surface $Y$ over $k$ such that $C = \pi^{-1}(P)$ for some closed point $P \in Y$. Prove that $C^2 < 0$. Hint: pull back a divisor of $Y$ containing $P$ and another divisor not containing $P$.

6. Using Castelnuovo’s criterion, give an alternate proof that the blowup of $\mathbb{P}_k^2$ at two points is isomorphic to the blowup of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at one point.