Math 203C (Algebraic Geometry), UCSD, spring 2013
Problem Set 6 (due Wednesday, May 29)

Solve the following problems, and turn in the solutions to *four* of them. No homework due Wednesday, May 22 due to qualifying exams.

1. Let $f : Y \to X$ be a finite surjective morphism between integral noetherian schemes. Let x, y be the generic points of X, Y. Let $\mathcal{L}$ be a line bundle on Y, let U be a neighborhood of x, let g be a global section of $\mathcal{L}$, and suppose that $y \in Y_g \subseteq f^{-1}(U)$. Prove that for any sufficiently large $n > 0$, there exists a homomorphism $u : \mathcal{O}_X^m \to f_*(\mathcal{L}^{\otimes n})$ of $\mathcal{O}_X$-modules for some $m > 0$ which is an isomorphism over some neighborhood of x.

2. Let $f : Y \to X$ be a finite surjective morphism of proper integal schemes over a field k. Let $\mathcal{L}$ be a line bundle on X. Prove that if $f^*\mathcal{L}$ is ample, then so is $\mathcal{L}$. Hint: use the previous exercise to reduce from X to a closed subscheme of lower dimension.

3. Let k be an algebraically closed field. Let X be the blowup of $\mathbb{P}^2_k$ at five distinct closed points $P_1, \dots, P_5$, no three of which are collinear, viewed as a degree 4 Del Pezzo surface in $\mathbb{P}^4_k$ (via the linear system $|3L - P_1 - \cdots - P_5|$). Prove that X contains exactly 16 lines of $\mathbb{P}^4_k$.

4. Let f be a rational function on a smooth projective connected surface X over an algebraically closed field k. Prove that there exists a morphism $\pi : \tilde{X} \to X$ which is a composition of monoidal transformations such that $\pi^*(f)$ defines a morphism $\tilde{X} \to \mathbb{P}^1_k$. Hint: the key point is to separate the zero locus and the pole locus.

5. Let C be an irreducible curve on a smooth projective connected surface X over an algebraically closed field k. Suppose that there exists a morphism $\pi : X \to Y$ to a projective (but not necessarily smooth) surface Y over k such that $C = \pi^{-1}(P)$ for some closed point $P \in Y$. Prove that $C^2 < 0$. Hint: pull back a divisor of Y containing P and another divisor not containing P.

6. Using Castelnuovo's criterion, give an alternate proof that the blowup of $\mathbb{P}^2_k$ at two points is isomorphic to the blowup of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at one point.