Math 203C (Algebraic Geometry), UCSD, spring 2013
Solutions for problem set 7

1. The map $\mathrm{GL}(m) \times_{\mathrm{Spec}\,\mathbb{Z}} Z_J \to V_J$ is the multiplication map on functors of points. The map $V_J \times \mathrm{GL}(m) \times_{\mathrm{Spec}\,\mathbb{Z}} Z_J$ is defined on functors of points to send a matrix $A = (x_{ij})$ to the pair consisting of $B = (x_{i,i_\ell})$ and $B^{-1}A$.

2. First, note that switching two terms in either I or J negates the relation $Q_{I,J,s}$. In particular, if $x_I = 0$ or $x_J = 0$ then $Q_{I,J,s} = -Q_{I,J,s}$ and so $Q_{I,J,s} = 0$. We may thus assume $x_I, x_J \neq 0$ hereafter.

   Next, note that it is enough to check that each $f^*(Q_{I,J,s})$ vanishes. Using the previous paragraph, we may reduce to the case $J = (1, \dots, m)$.

   Next, note that since $\mathbb{A}^{mn}_{\mathbb{Z}}$ is irreducible, we need only check that $f^*(Q_{I,J,s})$ vanishes on the open dense subscheme U_J.

   Next, note that f is $\mathrm{GL}(m)$-equivariant and $U_J \cong \mathrm{GL}(m) \times_{\mathrm{Spec}\,\mathbb{Z}} Z_J$, so it suffices to check that $f^*(Q_{I,J,s})$ vanishes on Z_J. Now we are checking the vanishing of

$$\det(x_{i,i_\ell}) - \sum_{t=1}^{m} f^*(x_{(i_1,\dots,i_{s-1},j_t,i_{s+1},\dots,i_m)})x_{t,i_s},$$

   but $f^*(x_{(i_1,\dots,i_{s-1},j_t,i_{s+1},\dots,i_m)})$ can be interpreted as the signed (s,t)-minor of the matrix $\det(x_{i,i_\ell})$. This proves the claim.

3. We may again assume without loss of generality that $J = (1, \dots, m)$. In this case, the inverse map $g : G(m,n)_{x_J} \to Z_J$ may be constructed as

$$g^*(x_{hk}) = x_{(1,\dots,h-1,k,h+1,\dots,m)}/x_J.$$

   It is clear (from looking at functors of points) that $g \circ f$ is the identity on Z_J. To check that $f \circ g$ is the identity, again we look at functors of points. Take R-valued points for any ring R. Given a point of $\mathbb{P}^{\binom{n}{m}-1}$ satisfying the Plücker relations, we may write x_I in terms of the $g^*(x_{hk})$ and the $x_{I'}$ where I' is obtained from I by replacing one element with an element of $\{1, \dots, m\}$. By repeating this construction, we eventually write x_I in terms of the $g^*(x_{hk})$ and nothing else. This proves that each point satisfying the Plücker relations is uniquely determined by the $g^*(x_{hk})$, from which it follows that $f \circ g$ is the identity.

4. (a) It suffices to take the intersection of all closed subschemes of X through which f factors. Since X is noetherian, this ends being a finite intersection, so is again a closed subscheme.

   (b) Since S is affine and $X \to S$ is quasicompact, X is quasicompact. We may thus cover X by finitely many open affine subschemes $U_1, \dots, U_n$. Since $X \to S$ is locally of finite type, each U_i corresponds to a finitely generated algebra over the coordinate ring of S, and hence admits a closed immersion into $\mathbb{A}^n_S$. Embed $\mathbb{A}^n_S$ into $\mathbb{P}^n_S$ and then let P_i be the closed image of $U_i \to \mathbb{P}^n_S$.

(c) If X is irreducible, we may replace X with the disjoint union of its components and build the map on top of this. So let us assume X is irreducible. Define $U_1, \dots, U_n$ as in (b), excluding any empty subschemes; then $U = U_1 \cap \cdots \cap U_n$ is open and nonempty, hence dense. Let X' be the closed image of $U \to X \times_S P$ for $P = P_1 \times_S \cdots \times_S P_n$. To see that this works, note first that $U \to X \times_S P$ is an open immersion, so $X' \times_X U \to U$ is an isomorphism. Then note that $X \times_S P \to P$ is the base extension of $X \to S$ and hence is proper, so $X' \to X \times_S P \to P$ is both proper and an immersion, hence a closed immersion. Since P is projective over S (via a Segre embedding), so is X'.

5. (a) The local ring $\mathcal{O}_{T,P}$ has fraction field $K(T)$ and is integrally closed therein. Let R be the integral closure of $\mathcal{O}_{T,P}$ in $K(C)$; then R is the direct sum of the local rings of C over the preimages of P. If these separate into multiple G-orbits, then summing each orbit gives a nontrivial direct sum decomposition of R into G-stable summands. The projection onto one such summand is then a G-invariant idempotent element of R other than 0 and 1, but no such element exists because R^G equals $R \cap K(T) = \mathcal{O}_{T,P}$ (because $\mathcal{O}_{T,P}$ is integrally closed in $K(T)$) and this ring is connected.

(b) Since we are in characteristic 0, we may compute ramification numbers by counting missing preimages. For each closed point P in T, the preimages of P are permuted transitively (by Galois theory), so each one makes the same contribution to the ramification number so the number of such preimages is n/s_P where s_P is the order of the stabilizer of any one point in the preimage. We then have $r_P = s_P$ and by Riemann-Hurwitz,

$$2g - 2 = n(2g(T) - 2) + \sum_P \left(n - \frac{n}{r_P} \right).$$

(c) Write

$$C = 2h - 2 + \sum_{i=1}^m \left(1 - \frac{1}{r_i} \right).$$

If $h \geq 2$, then $C \geq 2$. If $h = 1$, then if $m = 0$ we have $C = 0$, and if $m \geq 1$ we have $C \geq 1/2$. We may thus assume $h = 0$ hereafter.

If $m \leq 2$, then $C < 2$. If $m \geq 5$, then $C \geq 5/2$. We thus need only consider the cases $m = 3$ and $m = 4$. We may assume $2 \leq r_1 \leq r_2 \leq \cdots$.

Suppose $m = 4$. If $r_3 \geq 3$, then $C \geq -2 + 1/2 + 1/2 + 2/3 + 2/3 = 1/3$. The only other cases are those with $r_1 = r_2 = r_3 = 2$, in which case $C = 0$ for $r_4 = 2$ and $C \geq 1/6$ for $r_4 \geq 3$.

Suppose $m = 3$. If $r_1 \geq 3$, then $C = 0$ for $r_3 = 3$ and $C \geq 1/12$ for $r_3 \geq 4$. Otherwise, we have $r_1 = 2$. In this case:

- If $r_2 = 2$ then $C < 0$.

- If $r_2 = 3$, then $C \leq 0$ for $r_3 \leq 6$ and $C \geq 1/42$ for $r_3 \geq 7$.
- If $r_2 = 4$, then $C = 0$ for $r_3 = 4$ and $C \geq 1/20$ for $r_3 \geq 5$.
- If $r_2 \geq 5$, then $C \geq -2 + 1/2 + 4/5 + 4/5 = 1/10$.

(d) Since $g \geq 2$, $(2g-2)/n > 0$. By (b) and (c), we must then have $(2g-2)/n \geq 1/42$, so $n \leq 84(g-1)$.

6. (a) The map f has degree $2g - 2 = 2$ (by Riemann-Roch) and is ramified at 6 points (by Riemann-Hurwitz). Let $g : C \to C$ be an automorphism fixing the 6 ramification points; it must then fix all of $\mathbb{P}^1_k$ since a nontrivial automorphism of $\mathbb{P}^1_k$ can only fix 2 points. If we write C as the curve $y^2 = P(x)$ with P a rational function of degree 6, then g acts fixing x, so g is either the identity or an involution. In particular, if g is of odd order, it must be the identity.

   (b) If the Hurwitz bound were achieved by C, then C would have a group of automorphisms of order 84 and hence an automorphism of order 7. But this automorphism would then act as a nontrivial permutation of 6 ramification points, a contradiction.

7. (a) The partial derivatives of $x^3y+y^3z+z^3x$ in x, y, z are respectively $3x^2y+z^3, 3y^2z+x^3, 3z^2x+y^3$. For these to all vanish, we must have $x^3 = -3y^2z, y^3 = -3z^2x, z^3 = -3x^2y$. Multiplying these together gives $(xyz)^3 = -27(xyz)^3$, so $xyz = 0$. But if any one of x, y, z vanishes then they must all do so: if for instance $x = 0$, then $y^3 = -3z^2x$ and $z^3 = -3x^2y$ so we must also have $y = z = 0$. This proves that the curve is smooth, and its genus must then be $\binom{4-1}{2} = 3$.

   (b) For ζ_7 a primitive 7-th root of unity, take

$$[x : y : z] \mapsto [\zeta_7 x : \zeta_7^4 y : \zeta_7^2 z].$$

   To get divisibility by 3, we exhibit an automorphism of order 3:

$$[x : y : z] \mapsto [y : z : x].$$

   (c) Let G be the group of automorphisms of C. By (b) and our assumptions, G has order divisible by 2^3, 3, and 7, and hence by $168 = 84(g-1)$. Hence we must have equality in the Hurwitz bound.

8. The partial derivatives of $y^{q+1} - zx^q - z^q x$ in x, y, z are respectively $-z^q, y^q, -x^q$, which have no common zero. Hence this curve is smooth, so its genus is $\binom{q+1-1}{2} = q(q-1)/2$. Meanwhile, the curve has automorphisms by the additive group $\mathbb{F}_q$ via

$$[x : y : z] \mapsto [x + cz : y : z] \qquad (c \in \mathbb{F}_q)$$

and by the group of $(q+1)$-st roots of unity in k via

$$[x : y : z] \mapsto [x : cy : z] \qquad (c \in k, c^{q+1} = 1).$$

These together form a group of order $q(q+1)$, which is not yet enough to get the contradiction. However, we can do better: for $c \in \mathbb{F}_{q^2}^*$, we have automorphisms

$$[x : y : z] \mapsto [c^{q+1}x : cy : z].$$

Now we get a group of order $q(q^2-1)$, which is bigger than $84(g-1)q(q-1)/2$ for q sufficiently large.