**Math 203C (Algebraic Geometry), UCSD, spring 2013**
**Problem Set 7 (due Wednesday, June 5)**

Solve the following problems, and turn in the solutions to *four* of them, including at most two of 1–3 and at most one of 7–8.

Notation for problems 1–3: fix positive integers $n \geq m > 0$. Put

$$\mathbb{A}_{\mathbb{Z}}^{mn} = \operatorname{Spec} \mathbb{Z}[x_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n]$$

and

$$\mathbb{A}_{\mathbb{Z}}^{\binom{n}{m}} = \operatorname{Spec} \mathbb{Z}[x_J : J = (j_1, \dots, j_m), 1 \leq j_1 < \cdots < j_m \leq n].$$

For $I = (i_1, \dots, i_m) \in \{1, \dots, n\}^m$, put

$$x_I = \begin{cases} 0 & \text{if } i_1, \dots, i_m \text{ are not all distinct} \\ \operatorname{sgn}(\sigma) x_J & \text{if } I = \sigma(J) \text{ and } j_1 < \cdots < j_m. \end{cases}$$

Define the map $f : \mathbb{A}_{\mathbb{Z}}^{mn} \to \mathbb{A}_{\mathbb{Z}}^{\binom{n}{m}}$ by

$$f^*(x_J) = \det(x_{i,j_\ell} : 1 \leq i, \ell \leq m).$$

Let $V_J$ be the basic open subscheme of $\mathbb{A}_{\mathbb{Z}}^{mn}$ defined by $f^*(x_J)$. Let $Z_J$ be the closed subscheme of $\mathbb{A}_{\mathbb{Z}}^{mn}$ (and $V_J$) defined by the relations $(x_{i,j_\ell}) - I_m$.

1. Prove that $V_J \cong \mathrm{GL}(m) \times_{\operatorname{Spec} \mathbb{Z}} Z_J$. Hint: you should be using Yoneda's lemma throughout this problem set, in order to express such questions in terms of points over affine schemes.

2. Define the *Plücker relations* as follows: for $I, J \in \{1, \dots, n\}^m$ and $s \in \{1, \dots, m\}$, put

$$Q_{I,J,s} = x_I x_J - \sum_{t=1}^{m} x_{(i_1, \dots, i_{s-1}, j_t, i_{s+1}, \dots, i_m)} x_{(j_1, \dots, j_{t-1}, i_s, j_{t+1}, \dots, j_m)}.$$

   Prove that $f$ factors through the closed subscheme of $\mathbb{A}_{\mathbb{Z}}^{\binom{n}{m}}$ cut out by the $Q_{I,J,s}$. Hint: first reduce to considering only $Z_J$.

3. Let $G(m,n)$ be the closed subscheme of $\mathbb{P}^{\binom{n}{m}-1}$ cut out by the $Q_{I,J,s}$. By the previous exercise, $f$ induces a map $Z_J \to G(m,n)_{x_J}$. Prove that this map is an isomorphism.

4. In this exercise we prove a form of *Chow's lemma*, which we used in class a while back.

   (a) For any morphism $f : Y \to X$ of schemes with $X$ noetherian, prove that there is a minimal closed subscheme $Z$ of $X$ through which $f$ factors. We call this the *closed image* of $f$.

(b) Let $S$ be a noetherian affine scheme. Let $X \to S$ be a morphism of finite type. Prove that $X$ can be covered by finitely many open subschemes $U_1, \dots, U_n$ which are *quasiprojective* over $S$. That is, each $U_i$ admits an open immersion (over $S$) into a projective $S$-scheme $P_i$. Hint: embed affine space into projective space.

(c) Let $X \to S$ be a proper morphism of schemes with $S$ affine and noetherian. Prove that three exists a morphism $X' \to X$ with $X' \to S$ proper such that for some open dense subscheme $U$ of $X$, the map $X' \times_X U \to U$ is an isomorphism. Hint: reduce to the case of $X$ irreducible. Then put $U = U_1 \cap \cdots \cap U_n$ and let $X'$ be the closed image of $U \to X \times_S P_1 \times_S \cdots \times_S P_n$.

5. Let $k$ be an algebraically closed field of characteristic 0. Let $C$ be a smooth projective connected curve over $k$ of genus $g \geq 2$. In this exercise, we prove that any finite group $G$ of automorphisms of $C$ has order at most $84(g-1)$ (the *Hurwitz bound*). This bound is achieved in some cases, e.g., for $g = 3$ (see below). It can also be shown that any group of automorphisms of $C$ is finite, but we will not do this here.

(a) Let $n$ be the order of $G$; then the $G$-fixed subfield of $K(C)$ is itself the function field of a curve $T$, and the inclusion of fields corresponds to a morphism $f : C \to T$ of degree $n$. Prove that $G$ acts transitively on the preimage of any point of $T$.

(b) For each closed point $P \in T$, let $r_P$ denote the ramification number of the map at some preimage of $P$ (by (a), it does not matter which one is chosen). Prove that

$$\frac{2g-2}{n} = 2g(T) - 2 + \sum_P \left(1 - \frac{1}{r_P}\right).$$

Note that the sum is finite because $r_P = 1$ for all but finitely many $P$.

(c) Prove there do not exist integers $h \geq 0$ and $r_1, \dots, r_m \geq 2$ such that

$$0 < 2h - 2 + \sum_{i=1}^{m} \left(1 - \frac{1}{r_i}\right) < \frac{1}{42}.$$

(d) Deduce that $n \leq 84(g-1)$.

6. Let $C$ be a curve of genus 2 over an algebraically closed field of characteristic $\neq 2$.

(a) Let $f : C \to \mathbb{P}^1_k$ be the map defined by the canonical divisor. Prove that any finite group of automorphisms of $C$ of odd order acts faithfully on the ramification points of $f$.

(b) Prove that the Hurwitz bound is *never* achieved for $g = 2$.

7. Let $k$ be an algebraically closed field of characteristic 0. Let $C$ be the *Klein quartic* curve

$$\operatorname{Proj} k[x, y, z]/(x^3 y + y^3 z + z^3 x).$$

(a) Prove that $g(C) = 3$.

(b) Write down explicit automorphisms of $C$ of orders 3 and 7.

(b) It can be shown that the group of automorphisms of $C$ is finite and contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Assuming these facts, prove that $C$ achieves the Hurwitz bound.

8. Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $q$ be a power of $p$. Prove that

$$\operatorname{Proj} k[x, y, z]/(y^{q+1} - zx^q - z^q x)$$

is a smooth projective curve which violates the Hurwitz bound for $q$ sufficiently large compared to $p$.