Math 203C (Algebraic Geometry), UCSD, spring 2013
Notes on smooth morphisms

Although I originally introduced the Stacks Project definition of smoothness, I'm now going to use the EGA definitions as introduced on PS 1 and leave the proof of their equivalence to the Stacks Project. (The precise EGA reference is EGA 4 section 17, but you shouldn't need to refer to this anytime soon.)

Let $f : Y \to X$ be a morphism of schemes. We say f is *formally smooth/unramified/étale* if for every affine scheme $X' = \mathrm{Spec}(R)$ and every closed subscheme X'_0 of X defined by a nilpotent ideal of R, every diagram

$$\begin{array}{ccc} X'_0 & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ X' & \longrightarrow & X \end{array}$$

admits at least/at most/exactly one extension by a dashed arrow. We say that the morphism f is *smooth/unramified/étale* if f is formally smooth/unramified/étale and locally of finite presentation.

Lemma 1. *All of these properties are local on the source and on the target.*

Proof. Homework. □

Lemma 2. *The morphism f is formally unramified if and only if $\Omega_{Y/X} = 0$.*

Proof. Homework. □

Lemma 3. *Let $R \to S$ be a formally smooth morphism of rings. Let I be an ideal of S such that $R \to T = S/I$ is also formally smooth. Then I/I^2 is a projective T-module.*

Proof. Because $R \to T$ is formally smooth, the isomorphism $T \to S/I$ factors through S/I^2; that is, the exact sequence

$$0 \to I/I^2 \to S/I^2 \to T \to 0$$

of R-modules is split.

To check projectivity, we must show that for any surjection $P \to Q$ of T-modules, any morphism $I/I^2 \to Q$ factors through P. First, push out the previous exact sequence along $I/I^2 \to Q$ to obtain another exact sequence

$$0 \to Q \to E \to T \to 0$$

of R-modules which is again split by a ring homomorphism $T \to E$. Then use $P \to Q$ to make a surjection $T \oplus P \to T \oplus Q \cong E$. The kernel of $T \oplus P \to E$ is contained in P, and so has square zero. Since $R \to S$ is formally smooth, we may factor $S \to S/I^2 \to E$ through $S \to T \oplus P$. This map kills I^2 and so induces a map $S/I^2 \to T \oplus P$. The image of I/I^2 under this map lands in P, giving a map $I/I^2 \to P$; this proves projectivity. (Argument taken from EGA 4, 0.19.5.3). □

Lemma 4. *If $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a formally smooth morphism of affine schemes, then $\Omega_{S/R}$ is a projective S-module.*

Proof. If $R \to S$ is formally smooth, then so are $S \to S \otimes_R S$ and $R \to S \to S \otimes_R S$. Let I be the kernel of $S \otimes_R S \to S$; by (b), $I/I^2 \cong \Omega_{S/R}$ is a projective S-module. □

The converse is not true; see homework. In the smooth case, we get a converse by also keeping track of dimensions.

Lemma 5. *Let k be a field and let $k \to S$ be a morphism of finite type (equivalently of finite presentation since k is noetherian). Then the following are equivalent.*

(a) The morphism $k \to S$ is (formally) unramified.

(b) The morphism $k \to S$ is (formally) étale.

(c) The ring S is a direct sum of finitely many finite separable field extensions of k.

Proof. Homework. □

This means for instance that the purely inseparable field extension $\mathbb{F}_p(x^p) \to \mathbb{F}_p(x)$ is not formally unramified, that is, $\Omega_{\mathbb{F}_p(x^p)/\mathbb{F}_p(x)} \neq 0$. See homework for the explicit computation.

We say that a ring homomorphism $R \to S$ satisfies the *Jacobian criterion* at $\mathfrak{p} \in \mathrm{Spec}(S)$ if there exists a surjection $R[x_1, \dots, x_n] \to S$ with kernel I such that $I_\mathfrak{p}$ admits generators $(f_1, \dots, f_m)$ such that the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial x_j} \right)$$

over $\kappa(\mathfrak{p})$ has rank $n - m$ (i.e., the $m \times m$ subdeterminants generate the unit ideal).

Lemma 6. *Let R be a local ring. Then a finitely presented morphism $R \to S$ is smooth if and only if $\mathrm{Spec}(S)$ satisfies the Jacobian criterion at each point.*

Proof. It was proven on homework that if $R \to S$ satisfies the Jacobian criterion, then it is smooth (this argument can be thought of an algebro-geometric analogue of the implicit function theorem). In the opposite direction, we will prove in fact that *every* presentation $S \cong R[x_1, \dots, x_n]/I$ has the desired property. Put $P = R[x_1, \dots, x_n]$. Then the map

$$I \to P \to \Omega_{P/R}$$

induces a map

$$I/I^2 \to \Omega_{P/R} \otimes_P S.$$

We check that this map is injective by producing a left inverse $\Omega_{P/R} \otimes_P S \to I/I^2$. The data of this map is that of a map $\Omega_{P/R} \to I/I^2$ of P-modules, or equivalently an R-linear derivation $P \to I/I^2$ or automatically $P/I^2 \to I/I^2$.

Since $R \to S$ is formally smooth, the sequence

$$0 \to I/I^2 \to P/I^2 \to S \to 0$$

admits an R-algebra splitting $S \to P/I^2$ and hence an R-algebra isomorphism $P/I^2 \cong S \oplus I/I^2$. Using the P-algebra multiplication, we get an R-linear derivation $P/I^2 \to I/I^2$ of the desired form.

To sum up, $I/I^2 \to \Omega_{P/R} \otimes_P S$ is injective. Consequently, we may take $f_1, \dots, f_m$ to be elements of I lifting a basis of $I/I^2 \otimes_P \kappa(\mathfrak{p})$. □

Lemma 7. *Let $R \to S$ be a smooth morphism of rings. Then $R \to S$ is also flat.*

Proof. We may assume R is a local ring. By Lemma 6, S satisfies the Jacobian criterion at each point. In particular, S is locally the quotient of $R[x_1, \dots, x_n]$ by a *regular sequence.* To deduce flatness from this, we may reduce to the case where R is noetherian (because R is the union of its $\mathbb{Z}$-finitely generated subrings and $R \to S$ can be realized over some such subring). Since $R[x_1, \dots, x_n]$ is flat over R, we may proceed by induction using the following fact: if A is a local ring with residue field k, $u : M \to N$ is a homomorphism of A-modules, N is flat, and $u \otimes_A k$ is injective, then $\operatorname{coker}(u)$ is flat. (Namely, N is flat iff $\operatorname{Tor}^1(N, k) = 0$, and use the long exact sequence for Tor.) □

Lemma 8. *A morphism is étale if and only if it is flat and unramified.*

Proof. If a morphism is étale, it is smooth and unramified, and hence étale by Lemma 7. For the other implication, see homework. □