Math 203B: Algebraic Geometry
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Analytification and GAGA

1 Analytification (after SGA 1)

Recall that the category of locally ringed spaces includes both schemes and complex analytic spaces. The latter are characterized as spaces which locally admit a closed immersion into some open subspace in some $\mathbb{C}^n$, with the latter carrying the usual topology and the sheaf of holomorphic functions. Implicit in this construction is the following fact, which I will not prove here (it depends on the Weierstrass preparation theorem).

Theorem 1. *Let D denote the* closed *unit disc in $\mathbb{C}$. Let R be the stalk of the structure sheaf of $\mathbb{C}^n$ at D^n, i.e., the direct limit of $\Gamma(U, \mathcal{O})$ over neighborhoods U of D^n in $\mathbb{C}^n$. (Note that R may be identified with the subring of $\mathbb{C}[\![z_1, \dots, z_n]\!]$ consisting of series converging on some open polydisc of radii strictly greater than 1.) Then the ring R is noetherian.*

This means that locally, any closed immersion into the analytic space $\mathbb{C}^n$ is locally defined by a finitely generated ideal. It also makes it sense to talk about *coherent sheaves* on complex analytic spaces.

We would like to construct a functor from schemes locally of finite type over $\mathbb{C}$ to complex analytic spaces that takes $\mathbb{A}^n_{\mathbb{C}}$ to $\mathbb{C}^n$. To do this, we will take advantage of the fact that both types of objects belong to the same ambient category.

Let X be a scheme locally of finite type over $\mathbb{C}$. By an *analytification* of X, we will mean a complex analytic space X^{an} equipped with a morphism $X^{\mathrm{an}} \to X$ satisfying the following universal property: for every complex analytic space Y, every morphism $Y \to X$ of locally ringed spaces factors uniquely through X^{an} (in the category of locally ringed spaces, or equivalently in the category of complex analytic spaces). As usual, since this construction is defined from a universal property, an analytification is unique up to unique isomorphism if it exists.

For example, take $X = \mathbb{A}^n_{\mathbb{C}}$. We claim that $X^{\mathrm{an}} = \mathbb{C}^n$ has the desired property. The map $\mathbb{C}^n \to X$ arises by adjunction from the map from $\mathbb{C}[z_1, \dots, z_n]$ to the ring of entire power series in $z_1, \dots, z_n$. For any morphism $Y \to X$ of locally ringed spaces in which Y is a complex analytic space, we can pull back $z_1, \dots, z_n$ to elements of $\Gamma(Y, \mathcal{O})$, and we must check that any entire power series in these elements converges in $\Gamma(Y, \mathcal{O})$. For this, we may reduce to the case where Y admits a closed immersion into some polydisc; we may then lift $z_1, \dots, z_n$ to sections on that polydisc. We are thus reduced to an elementary exercise in complex analysis (proof omitted): if $z_1, \dots, z_n$ are power series in some variables $y_1, \dots, y_m$ which converge on the polydisc $|y_1|, \dots, |y_m| < 1$, then any entire power series $\sum c_{i_1,\dots,i_n} z_1^{i_1} \dots z_n^{i_n}$ evaluates to a power series in $y_1, \dots, y_m$ convergent on the unit polydisc.

More generally, if X is affine of finite type over $\mathbb{C}$, then it admits a closed immersion $j : X \to \mathbb{A}^n_{\mathbb{C}}$. If $Y \to X$ is a morphism of locally ringed spaces with Y a complex analytic space, then we may compose to get a map $Y \to \mathbb{A}^n_{\mathbb{C}}$ which then gives rise to a morphism

$f : Y \to \mathbb{C}^n$. The kernel of the induced map $\mathcal{O}_{\mathbb{C}^n} \to f_* \mathcal{O}_Y$ includes the inverse image ideal sheaf (from $\mathbb{A}^n_{\mathbb{C}}$ to $\mathbb{C}^n$) of the kernel of $\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}} \to j_* \mathcal{O}_X$; but this gives us a candidate for X^{an}, namely the closed immersion into $\mathbb{C}^n$ defined by this inverse image. In fact, I can replace "inverse image" with "pullback" because of the following fact.

Theorem 2. *The map $\mathbb{C}^n \to \mathbb{A}^n_{\mathbb{C}}$ of locally ringed spaces is flat. That is, for each $y \in \mathbb{C}^n$ mapping to $x \in \mathbb{A}^n_{\mathbb{C}}$, the ring homomorphism $\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}},x} \to \mathcal{O}_{\mathbb{C}^n,y}$ is flat.*

Proof. This is obvious for $n = 1$. The general case will be treated in an exercise. $\square$

Since analytifications are unique up to unique isomorphism, we can then glue to obtain analytifications of any scheme X locally of finite type over $\mathbb{C}$. Note that the map $X^{\mathrm{an}} \to X$ will again be flat, by base change from the case of affine space: for any closed immersion $Y \to X$ of schemes we have

$$Y^{\mathrm{an}} \cong Y \times_X X^{\mathrm{an}}$$

as a fiber product in the category of locally ringed spaces.

Aside: over a *nonarchimedean* complete field such as $\mathbb{Q}_p$, once one has in hand a suitable definition of analytic spaces over that field, one can similarly define analytifications of schemes locally of finite type over the field. However, there are several approaches to analytic geometry over a nonarchimedean field: Tate's rigid analytic spaces (equivalently, Raynaud's theory of formal schemes modulo admissible blowups), Berkovich's nonarchimedean analytic spaces, Huber's adic spaces...

2 Statement of the GAGA theorems

Suppose now that X is *projective* over $\mathbb{C}$. Then Serre proved the following statements about the relationship between X and its analytification. (As explained by Grothendieck in SGA1, one can promote these results to proper varieties over $\mathbb{C}$ using Chow's lemma, but we'll skip this refinement in the present discussion.)

Theorem 3 (GAGA)**.** *Let X be a projective scheme over $\mathbb{C}$. Let $\iota : X^{\mathrm{an}} \to X$ be the analytification morphism.*

(a) The pullback functor from coherent sheaves on X to coherent sheaves on X^{an} is an equivalence of categories.

(b) For every coherent sheaf $\mathcal{F}$ on X and every $i \geq 0$, the natural map $H^i(X, \mathcal{F}) \to H^i(X^{\mathrm{an}}, \iota^* \mathcal{F})$ is an isomorphism.

This statement has various geometric corollaries, notably the following.

Corollary 4. *Let X be a projective scheme over $\mathbb{C}$. Then every closed immersion into X^{an} arises from a closed immersion into X. For example, every analytic hypersurface in the complex-analytic projective n-space over $\mathbb{C}$ is in fact algebraic.*

Proof. For $j : Y \to X^{\mathrm{an}}$ a closed immersion, apply GAGA to lift $\ker(\mathcal{O}_{X^{\mathrm{an}}} \to j_* \mathcal{O}_Y)$ to an ideal subsheaf of $\mathcal{O}_X$. $\square$

3 Comparison of cohomology

We now sketch the proof of part (b) of the GAGA theorem. This follows the same general paradigm we have used to study sheaf cohomology, except that we need to draw on a couple of statements about complex analytic spaces which we will not take the time to prepare.

To begin with, note that as usual, we may replace a sheaf $\mathcal{F}$ on X with its pushforward along a closed immersion into a projective space. We may thus assume that $X = \mathbb{P}^n_{\mathbb{C}}$ for some n.

Let's next treat the case where $\mathcal{F} = \mathcal{O}$.

Input from complex analysis: by Liouville's theorem, the only bounded entire functions on $\mathbb{C}$ are the constants.

Using this input, we see that any bounded entire function on $\mathbb{C}^n$ is constant. In particular, $H^0(X^{\text{an}}, \mathcal{O}) = \mathbb{C} = H^0(X, \mathcal{O})$.

Input from complex analysis: let $U \subset \mathbb{C}^n$ be an open set which is the product of open discs. Then $H^i(U, \mathcal{O}) = 0$ for all $i > 0$. (This follows from a theorem of Cartan on *Stein spaces*.) The same then holds for $U = \mathbb{C}^n$ by taking direct limits.

This lets us Čech cohomology to compute $H^i(X^{\text{an}}, \mathcal{O})$ using the same cover that we use in algebraic geometry, and thus deduce that $H^i(X^{\text{an}}, \mathcal{O}) = 0 = H^i(X, \mathcal{O})$ for all $i > 0$.

Let's next treat the case where $\mathcal{F} = \mathcal{O}(m)$ for some $m \in \mathbb{Z}$, by induction on n. Take $n = 0$ as the trivial base case. Let $H \subset X$ be the hyperplane at infinity and let $j : H \to X$ be the canonical inclusion; there is then an exact sequence

$$0 \to \mathcal{O}_X(m-1) \to \mathcal{O}_X(m) \to j_*\mathcal{O}_H(m) \to 0.$$

For any given n, if we assume the claim for H (for all m), comparing long exact sequences in cohomology allows us to take the claim for X and extend it from a given m to either $m+1$ or $m-1$. Since we start with the single value $m = 0$ for which we do know the claim for X, we get it for all m.

Finally, we treat arbitrary $\mathcal{F}$ by dimension shifting, namely, we proceed by descending induction on i. Let's start with the base case $i > n$, for which we have $H^i(X, \mathcal{F}) = 0$ and so want the same for $H^i(X^{\text{an}}, \mathcal{F})$. We already know this if $\mathcal{F}$ is a direct sum of $\mathcal{O}(m)$'s. We might try to then use an exact sequence of the form

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is such a direct sum, but this is not apparently useful: we end up comparing $H^i(X^{\text{an}}, \iota^*\mathcal{F})$ to $H^{i+1}(X^{\text{an}}, \iota^*\mathcal{G})$, which again we know nothing about. The trick is that this doesn't last forever! If we form a projective resolution

$$\cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

then the Hilbert syzygy theorem can be used to truncate this sequence by replacing $\mathcal{E}_n$ with a direct summand, for which we also have the vanishing of H^i for $i > n$. This doesn't quite give the base case we were originally looking for, but it does at least give vanishing for $i > 2n$. All that really matters is that this bound is *independent of* $\mathcal{F}$.

Now to descend further, again cover $\mathcal{F}$ with a sheaf $\mathcal{E}$ which is a direct sum of various $\mathcal{O}(m)$, and then take apply cohomology to

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

to deduce vanishing of $H^i(X^{\mathrm{an}}, \iota^*\mathcal{F})$ from vanishing of $H^{i+1}(X^{\mathrm{an}}, \iota^*\mathcal{G})$.

4 Full faithfulness

Let us now start working on part (a) of the GAGA theorem. We will further break this up into two statements:

(i) The pullback functor is *fully faithful*: for any two coherent sheaves $\mathcal{F}, \mathcal{G}$ on X, the map $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(i^*\mathcal{F}, i^*\mathcal{G})$ is bijective. (This is itself really two statements; the map is injective, and it is surjective.)

(ii) The pullback functor is *essentially surjective*: every coherent sheaf on X^{an} is isomorphic to the pullback of some coherent sheaf on X.

We will work on full faithfulness first, since it requires less analytic input. In fact, it is an easy consequence of the comparison of cohomology (which is why we did that first).

The first point is that $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ can be interpreted as the global sections of the *sheaf Hom*, i.e., the sheaf $\mathcal{E}$ for which

$$\mathcal{E}(U) = \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{F}(U), \mathcal{G}(U)).$$

Moreover, forming the sheaf Hom commutes with analytification (because the analytification morphism is flat). So full faithfulness reduces to the fact that analytification preserves global sections, which is the case $i = 0$ of part (b). (Note also that again, even if we care only about global sections, we need to control higher cohomology to get access to them!)

5 Essential surjectivity

To finish proving part (a) of the GAGA theorem, we need to establish that any coherent sheaf on X^{an} is the pullback of a coherent sheaf on X. Again, we may assume $X = \mathbb{P}^n_{\mathbb{C}}$. We will proceed by induction on n, using $n = 0$ as a trivial base case.

To prove the claim for a given n, it would be enough to establish the analogue of Serre's theorem about generation: for every coherent sheaf $\mathcal{F}$ on X^{an}, there exists $m_0 \geq 0$ such that for every $m \geq m_0$ (or even just for $m = m_0$), the sheaf $\mathcal{F}(m)$ is generated by finitely many global sections. Namely, we then have a surjection $\iota^*\mathcal{E}_0 \to \mathcal{F}$, and apply the same argument to the kernel of this map to get an exact sequence

$$\iota^*\mathcal{E}_1 \to \iota^*\mathcal{E}_0 \to \mathcal{F} \to 0$$

where $\mathcal{E}_0, \mathcal{E}_1$ are coherent sheaves on X^{an}. By full faithfulness, the morphism $\iota^*\mathcal{E}_1 \to \iota^*\mathcal{E}_0$ is the pullback of a morphism $\mathcal{E}_1 \to \mathcal{E}_0$, and now $\mathcal{F} = \iota^* \operatorname{coker}(\mathcal{E}_1 \to \mathcal{E}_0)$ by the right exactness of pullback.

So now we are trying to generate $\mathcal{F}(m)$ by finitely many global sections. In fact, we don't need to worry about the "finitely many" part: if we get enough sections to generate the stalk at one point, then we also generate in a neighborhood by Nakayama's lemma, and the compactness of X^{an} does the job. By the same token, we need only generate the *fiber* at one point, not even the stalk.

Let's then work locally around a single point $y \in X^{\mathrm{an}}$, which corresponds to a closed point $x \in X$. Draw a hyperplane H through x and let $j : H \to X$ be the closed immersion. As usual, we have an exact sequence

$$0 \to \mathcal{O}_X(m-1) \to \mathcal{O}_X(m) \to j_*\mathcal{O}_H(m) \to 0$$

and likewise after pullback along ι. Tensoring with $\mathcal{F}$ doesn't preserve exactness at the left. Let $\mathcal{G}$ be the kernel of $\mathcal{F}(-1) \to \mathcal{F}$, so that we have an exact sequence

$$0 \to \mathcal{G}(m) \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to j_*j^*\mathcal{F}(m) \to 0.$$

This splits into two short exact sequences:

$$\begin{gathered} 0 \to \mathcal{G}(m) \to \mathcal{F}(m-1) \to \mathcal{H} \to 0 \\ 0 \to \mathcal{H} \to \mathcal{F}(m) \to j_*j^*\mathcal{F}(m) \to 0. \end{gathered}$$

We then get long exact sequences:

$$\begin{gathered} H^1(X^{\mathrm{an}}, \mathcal{F}(m-1)) \to H^1(X^{\mathrm{an}}, \mathcal{H}) \to H^2(X^{\mathrm{an}}, \mathcal{G}(m)) \\ H^1(X^{\mathrm{an}}, \mathcal{H}) \to H^1(X^{\mathrm{an}}, \mathcal{F}(m)) \to H^1(X^{\mathrm{an}}, j_*j^*\mathcal{F}(m)). \end{gathered}$$

Note that both $j_*j^*\mathcal{F}(m)$ and $\mathcal{G}(m)$ are actually pullbacks of coherent sheaves on H^{an}, so by the induction hypothesis we know they arise by pullback from H. We may thus apply part (b) of the GAGA theorem to deduce that $H^2(X^{\mathrm{an}}, \mathcal{G}(m)) = H^1(X^{\mathrm{an}}, j_*j^*\mathcal{F}(m)) = 0$ for m large. We now bring in one last piece of complex analysis, a theorem of Cartan (more on which if we have time).

Input from complex analysis: For any coherent sheaf $\mathcal{F}$ on X^{an} and any $i \geq 0$, $H^i(X^{\mathrm{an}}, \mathcal{F})$ is a finite-dimensional $\mathbb{C}$-vector space.

From this, we see that

$$\dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{F}(m-1)) \geq \dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{H}) \geq \dim_{\mathbb{C}}(X^{\mathrm{an}}, \mathcal{F}(m)).$$

That is, as m increases, the dimensions $\dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{F}(m))$ form a monotone nonincreasing sequence of nonnegative integers. Consequently, this sequence must stabilize at some value (which we know *a posteriori* will have to be 0, but we can't prove that just yet). Once we hit the stable value, we know for sure that

$$\dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{F}(m-1)) = \dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{H}) = \dim_{\mathbb{C}}(X^{\mathrm{an}}, \mathcal{F}(m)),$$

so if we back up one of those long exact sequences, we see that for m large,

$$H^0(X^{\text{an}}, \mathcal{F}(m)) \to H^0(X^{\text{an}}, j_*j^*\mathcal{F}(m)) \to 0$$

is exact. By the induction hypothesis, for m large, $j_*j^*\mathcal{F}(m)$ is generated by global sections, and hence it's also generated by the restrictions of global sections of $\mathcal{F}(m)$. This means in particular that the fiber of $\mathcal{F}(m)$ at x is generated by global sections for m large, and as noted earlier this suffices.