Math 203B: Algebraic Geometry
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Integral and normal schemes

1 Irreducible and integral schemes

A topological space is *irreducible* if it is nonempty and cannot be written as the union of two closed proper subsets. A scheme is *irreducible* if its underlying topological space is irreducible. This agrees with the definition for varieties.

A scheme X is *integral* if it is nonempty and every open affine subscheme of X is the spectrum of an integral domain. For instance, if R is an integral domain, then $\operatorname{Spec} R$ is integral. (Let η be the generic point of $\operatorname{Spec} R$; it has residue field $\operatorname{Frac} R$. For every distinguished open affine $D(f)$ of $\operatorname{Spec} R$, we have $\eta \in D(f)$ and the map $R_f \to \operatorname{Frac} R$ is injective. It follows that for any open affine subset $U = \operatorname{Spec} S$ of $\operatorname{Spec} R$, we have $\eta \in U$ and the map $S = \mathcal{O}_{\operatorname{Spec}(R)}(U) \to \mathcal{O}_{\operatorname{Spec}(R),\eta} = \operatorname{Frac} R$ is injective. Hence S is an integral domain.)

Note that an integral scheme is reduced, since the latter is a local property and holds for the spectrum of an integral domain. An integral scheme is also irreducible: otherwise, it would contain two disjoint open affine subschemes U_1, U_2, and then $U_1 \cup U_2$ would be affine with coordinate ring $\mathcal{O}(U_1) \oplus \mathcal{O}(U_2)$ which is not an integral domain.

Conversely, any scheme X which is reduced and irreducible is integral: every open affine subscheme $U = \operatorname{Spec} R$ of X must again be irreducible (if U is covered by two closed proper subsets, then those must be the intersections with U of closed sets Z_1, Z_2, and then $Z_1 \cup (X - U)$ and Z_2 form a covering of X by two proper closed subsets) and reduced. But now R cannot have any zero-divisors: if $f, g \in R$ satisfy $fg = 0$, then irreducibility implies that either $V(f)$ or $V(g)$ covers U; if $V(f) = U$ then f is nilpotent, and hence zero because U is reduced. Hence R is an integral domain.

Warning: if X is integral, then every local ring $\mathcal{O}_{X,x}$ is an integral domain. The converse fails trivially if X is not connected, but it can fail nontrivially even if X is connected; see Stacks Project tag 0568. Of course the problem is with irreducibility rather than reducedness. (I think that the converse is okay if X is locally noetherian, but I don't have the reference handy as I type this.)

2 Integral closure

For $R \subseteq S$ an inclusion of rings, the *integral closure* of R in S is an R-subalgebra of S; we say that R is *integrally closed* in S if the integral closure is equal to R itself.

If R is an integral domain, we say that R is *integrally closed* if it is integrally closed in its field of fractions. For example, any principal ideal domain has this property, as does any Dedekind domain. Also, any unique factorization domain is integrally closed. A typical example of a domain which is not integrally closed is $k[y, z]/(y^3 - z^2)$ for k an algebraically closed field; its integral closure in its fraction field is $k[x,y,z]/(y-x^2, z-x^3) \cong k[x]$.

This example has an important geometric meaning: if we put $X = \operatorname{Spec} k[y,z]/(y^3-z^2)$, $Y = \operatorname{Spec} k[x]$, then X is the affine part of a cuspidal cubic curve, Y is smooth, and $Y \to X$ is a finite morphism which becomes an isomorphism if we move the point $(y,z) \in X$ and the point $(x) \in Y$. That is, passing to the integral closure has removed a singularity!

To transfer this definition to schemes, it is better to avoid relying on integral domains. For R an arbitrary ring, the *total ring of fractions* of R is the localization of R at the multiplicative set consisting of all non-zero-divisors (i.e., all $f \in R$ for which $\times f : R \to R$ is injective). The map from R to its total ring of fractions is injective; we say that R is *integrally closed* if it is integrally closed in its total ring of fractions. Note that this doesn't force R to be reduced; for instance, if k is a field, then $k[x]/(x^2)$ coincides with its total ring of fractions, and is therefore reduced.

3 Normality

We say that a ring is *normal* if its local rings are integrally closed domains; in particular, a normal ring is reduced. An integrally closed reduced ring is normal, but in general not conversely; however, for noetherian rings the two definitions coincide (Stacks Project, tag 030C).

We now say that a scheme X is *normal* if for each $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain. Obviously, a ring R is normal iff $\operatorname{Spec} R$ is normal.

4 Normalization

For R a ring, let R' be the integral closure of R in its total ring of fractions. In general, R' is *integral* but not *finite* over R. It is easy to see the difference if we consider rings which are not noetherian; for instance, if

$$R = k[x_1^2, x_1^3, x_2^2, x_2^3, \dots]$$

then

$$R' = R[x_1, x_2, \dots]$$

is not finite over R.

However, the truth is in fact far more sinister: even if R is noetherian, it can happen that R' is not finite as an R-module! The first counterexample was constructed by Akizuki; see arXiv:alg-geom/9503017 for a modern exposition.

This strikes a severe blow against the idea of using noetherian hypotheses in algebraic geometry as a way to cut out all possible pathologies of commutative algebra. A better way was proposed by Grothendieck, based on the work of the "Japanese school" (Akizuki, Nakayama, Nagata, etc.): it is the notion of an *excellent ring* (and correspondingly an *excellent scheme*). I won't introduce the definition (see Stacks Project, Tag 07QS); I'll just say that on one hand, the class of excellent rings contains many basic examples (e.g., every field, plus the ring $\mathbb{Z}$) and is stable under many standard constructions (polynomial rings, quotients, completions, localizations at primes), and on the other hand, the integral closure of an excellent ring is finite.

In any case, one can define the *normalization* of X as a scheme X' over X such that $X' \to X$ factors uniquely through any finite morphism $Y \to X$ which is an isomorphism over some open dense subscheme of X. For example, if $R = \operatorname{Spec} k[x,y]/(xy)$ is the union of two crossing lines, then its normalization is the disjoint union

$$\operatorname{Spec}(k[x,y]/(x)) \oplus (k[x,y]/(y)).$$

In light of the above discussion, we can't say that the canonical morphism $X' \to X$ is finite except under some conditions more restrictive than noetherian, e.g., that X is a variety over an algebraically closed field.

We've seen an example of a nonsmooth irreducible variety whose normalization is smooth. However, don't take this to mean that normalization can remove all singularities! For example, if

$$R = k[x,y,z,w]/(xy - zw),$$

then $\operatorname{Spec} R$ is already normal, but is not smooth at $(0,0,0,0)$.