Solve the following problems, and turn in the solutions to four of them.

1. Let $k$ be a field. Prove that the ring $R = k[x, y, z, w]/(xy - zw)$ is an integrally closed domain, but does not correspond to a smooth variety.

2. Prove that a morphism of schemes which is both proper and affine is finite.

3. Let $R \to S$ be a homomorphism of rings of characteristic $p > 0$. Suppose that the Frobenius homomorphism $x \to x^p$ on $S$ is surjective. Prove that $\Omega_{S/R} = 0$.

4. (a) Let $X$ be the topological space $\mathbb{C}^n$ equipped with the sheaf $\mathcal{O}_X$ of holomorphic functions. Check that $\mathcal{O}_{X,x}$ is a locally ringed space. (All you need to know about holomorphic functions is that on any product of open discs, they are the functions defined by convergent Taylor series.)

(b) Let $X$ be the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation identifying $v$ with $cv$ for all $c \in \mathbb{C} - \{0\}$. Equip $X$ with a sheaf of rings in such a way that for $i = 0, \ldots, n$, the subset $\{[x_0 : \cdots : x_n] \in X : x_i \neq 0\}$ is isomorphic to the locally ringed space $\mathbb{C}^n$ of (b) via the map

$$[x_0 : \cdots : x_n] \mapsto \left( \frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i} \right).$$

Use the adjoint property of Spec to produce an injective map $X \to \mathbb{P}^n_{\mathbb{C}}$ of locally ringed spaces. This is an example of the process of analytification relating complex algebraic and analytic varieties, which we will use to study the GAGA theorem. (The case $n = 1$ was in a 203B homework.)

5. Find the flaw in this reasoning: for $k$ a field, an infinite disjoint union of copies of $\text{Spec}(k)$ is locally of finite type, but an infinite direct sum of copies of $k$ is not finitely generated as an $k$-algebra, contradiction.

6. Here is a result we will need for the GAGA theorem. Let $R$ be the subring of $\mathbb{C}[x_1, \ldots, x_n]$ consisting of those series which converge absolutely on some neighborhood of $(0, \ldots, 0)$. Using the Weierstrass preparation theorem, one can show that $R$ is noetherian (you do not need to provide a proof, but I recommend looking it up). Prove that $R$ is faithfully flat over $\mathbb{C}[x_1, \ldots, x_n](x_1, \ldots, x_n)$. Hint: compare both rings to $\mathbb{C}[x_1, \ldots, x_n]$. 