Math 203C (Algebraic Geometry), UCSD, spring 2016
Problem Set 1 (due Wednesday, April 6 by 5pm)

Solve the following problems, and turn in the solutions to *four* of them.

1. Let k be a field. Prove that the ring $R = k[x, y, z, w]/(xy - zw)$ is an integrally closed domain, but does not correspond to a smooth variety.

2. Prove that a morphism of schemes which is both proper and affine is finite.

3. Let $R \to S$ be a homomorphism of rings of characteristic $p > 0$. Suppose that the Frobenius homomorphism $x \to x^p$ on S is surjective. Prove that $\Omega_{S/R} = 0$.

4. (a) Let X be the topological space $\mathbb{C}^n$ equipped with the sheaf $\mathcal{O}_X$ of holomorphic functions. Check that $\mathcal{O}_{X,x}$ is a locally ringed space. (All you need to know about holomorphic functions is that on any product of open discs, they are the functions defined by convergent Taylor series.)

   (b) Let X be the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation identifying v with cv for all $c \in \mathbb{C} - \{0\}$. Equip X with a sheaf of rings in such a way that for $i = 0, \dots, n$, the subset $\{[x_0 : \cdots : x_n] \in X : x_i \neq 0\}$ is isomorphic to the locally ringed space $\mathbb{C}^n$ of (b) via the map

   $$[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right).$$

   Use the adjoint property of Spec to produce an injective map $X \to \mathbb{P}^n_{\mathbb{C}}$ of locally ringed spaces. This is an example of the process of *analytification* relating complex algebraic and analytic varieties, which we will use to study the GAGA theorem. (The case $n = 1$ was in a 203B homework.)

5. Find the flaw in this reasoning: for k a field, an infinite disjoint union of copies of Spec(k) is locally of finite type, but an infinite direct sum of copies of k is not finitely generated as an k-algebra, contradiction.

6. Here is a result we will need for the GAGA theorem. Let R be the subring of $\mathbb{C}[\![x_1, \dots, x_n]\!]$ consisting of those series which converge absolutely on some neighborhood of $(0, \dots, 0)$. Using the Weierstrass preparation theorem, one can show that R is noetherian (you do not need to provide a proof, but I recommend looking it up). Prove that R is faithfully flat over $\mathbb{C}[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$. Hint: compare both rings to $\mathbb{C}[\![x_1, \dots, x_n]\!]$.