Math 203C (Algebraic Geometry), UCSD, spring 2016
Problem Set 2 (due Wednesday, April 20 by 5pm)

Solve the following problems, and turn in the solutions to *four* of them.

1. Let X be a smooth, projective, irreducible curve over an algebraically closed field. Prove that a line bundle $\mathcal{L}$ on X is ample if and only if $\deg(\mathcal{L}) > 0$.

2. Let X be a closed subscheme of $\mathbb{P}^d_k$ for some field k.

   (a) Prove that a line bundle $\mathcal{L}$ on X is ample if and only if for each $m \in \mathbb{Z}$, there exists $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$, $\mathcal{L}^{\otimes n}(m)$ is generated by global sections. (Note that m stays fixed while n moves!)

   (b) Using (a), prove that if $\mathcal{L}$ is an ample line bundle on X, then its pullback to any closed subscheme of X is again ample. (Note that this is not immediately obvious from the definition of ampleness.)

3. Let X be a noetherian scheme, let U be an open subset, and let $\mathcal{F}$ be a coherent sheaf on U. Prove that $\mathcal{F}$ is the restriction of some (nonunique) coherent sheaf on X. See Hartshorne, Exercise 5.15 for the steps.

4. Let k be a field. Compare the blowups of $\mathbb{A}^2_k$ at the following ideals:

   $$(x, y), \qquad (x^2, y), \qquad (x, y^2), \qquad (x^2, y^2), \qquad (x^2, xy, y^2)$$

   to see which ones are isomorphic.

5. Let k be an algebraically closed field. Prove that the blowup of a smooth variety over k along a smooth subvariety is again smooth.

6. Let k be an algebraically closed field. Show that the blowup of $\mathbb{A}^2_k$ along the singular cubic curve $y^2 = x^3 - x^2$ is not smooth.

7. Do Hartshorne exercise II.7.12.