Math 203C (Algebraic Geometry), UCSD, spring 2016
Problem Set 3 (due Wednesday, May 4 by 5pm)

Solve the following problems, and turn in the solutions to *four* of them.

1. Prove that a flat morphism of finite presentation is open (that is, the image of every open set is open). You may assume Chevalley's theorem: the image of a morphism of finite presentation between affine schemes is constructible (not "open" as was typed in a previous version). (Hint: use the going-down theorem to prove that the image is stable under generalization.)

2. Prove that the morphism

$$\operatorname{Spec} k[x, y, z, w]/((z, w) \cap (x + z, y + w)) \to \operatorname{Spec} k[x, y]$$

is not flat.

3. Prove Chow's lemma as described in Hartshorne exercise II.4.10.

4. Let $f : X \to S$ be a proper morphism between noetherian schemes. Using Chow's lemma, prove that for any coherent sheaf $\mathcal{F}$ on X, the higher direct images $R^i f_* \mathcal{F}$ are coherent sheaves on S. Hint: induct on the dimension of the support of $\mathcal{F}$. If you get stuck, see Stacks Project, tag 02O3.

5. Prove that Stein factorization remains true without noetherian hypotheses: if $f : Y \to X$ is a proper morphism of arbitrary schemes and $f_* \mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism, then f has connected fibers. (Hint: assume X is affine, realize f over a finitely generated $\mathbb{Z}$-subalgebra of $\mathcal{O}(X)$, then add more generators to enforce the isomorphism $f_* \mathcal{O}_Y \to \mathcal{O}_X$.)

6. Let $Z \to X$ be a closed immersion of schemes (which you may assume to be noetherian if you wish). Prove that the blowup of X along Z is not flat over X unless it is isomorphic to X.

7. Let X be a scheme of finite type over an algebraically closed field k. Prove that the function $x \mapsto \dim_k \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ on the set of closed points of X is upper semicontinuous.