Math 203C (Algebraic Geometry), UCSD, spring 2016
Problem Set 4 (due Wednesday, May 25 by 5pm)

Solve the following problems, and turn in the solutions to *four* of them.

1. Let R be a noetherian ring and let M be a finitely generated R-module. We say M has *projective dimension* m if m is the smallest nonnegative integer for which there exists a projective resolution
$$\cdots \to P_1 \to P_0 \to M \to 0$$
of R-modules in which the P_i are finite projective and $P_i = 0$ for all $i > m$. Prove that the following conditions are equivalent.

   (i) The R-module M has projective dimension $\leq m$.

   (ii) We have $\mathrm{Ext}^{m+1}_R(M, N) = 0$ for all R-modules M.

   (iii) We have $\mathrm{Ext}^i_R(M, N) = 0$ for all R-modules N and all $i \geq m + 1$.

   (iv) For any projective resolution of M as above,
$$0 \to \mathrm{coker}(P_{m+1} \to P_m) \to P_{m-1} \to \cdots \to P_0 \to M \to 0$$
is also a projective resolution of M.

   Here by $\mathrm{Ext}^i_R(M, N)$, I mean the right derived functors of the contravariant functor $\mathrm{Hom}_R(\bullet, N)$. (They happens to coincide with the right derived functors of the covariant functor $\mathrm{Hom}_R(M, \bullet)$ but you shouldn't need to use this.)

2. Let $R = \mathbb{C}[x^2, x^3] \subset \mathbb{C}[x]$. Give an example of a finitely generated R-module which does not admit a finite projective resolution.

3. Using Chow's lemma, prove that both parts of the GAGA theorem hold for proper varieties over $\mathbb{C}$.

4. (a) Prove, or give a reference for, the following statement: for R an integral domain, the integral closure of R in $\mathrm{Frac}(R)$ equals the intersection of the valuation rings of $\mathrm{Frac}(R)$ containing R.

   (b) Use this to deduce that for any strongly rational convex polyhedral cone σ, the scheme $U_\sigma = \mathrm{Spec}\,\mathbb{C}[S_\sigma]$ is normal. (See notes from Wednesday, May 11.)

   (c) Give an example of a cone σ for which $U_\sigma \cong \mathrm{Spec}\,\mathbb{C}[x, y, z, w]/(xy - zw)$ (thus recovering an example from a previous homework).

5. Using toric varieties, exhibit a blowup of $\mathbb{P}^2_{\mathbb{C}}$ and a blowup of $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ which are isomorphic to each other.

6. Show that every 2-dimensional toric variety over $\mathbb{C}$ admits a blowup which is smooth over $\mathbb{C}$.