Math 204A (Number Theory), UCSD, fall 2020
Notes on extension of valuations

Let K be a field complete with respect to a nontrivial multiplicative valuation (absolute value) denoted $|\cdot|_K$. Let L be a finite extension of K. In these notes, we construct a multiplicative valuation on L extending the one on K without separating into archimedean and nonarchimedean cases. This uses some ideas borrowed from functional analysis.

Let S be the set of functions $|\cdot| : L \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in L$,

(a) $|0| = 0$, $|1| \geq 1$;

(b) $|x + y| \leq |x| + |y|$;

(c) $|xy| \leq |x||y|$;

(d) $|xy| = |x||y|_K$ when $y \in K$.

This set is nonzero because I can construct such a function by taking the supremum norm with respect to some K-basis. (The third condition will initially come out as $|xy| \leq c|x||y|$ for some fixed $c > 0$, but we can rescale to ensure that $c = 1$ works.)

Using Zorn's lemma, we can choose a *minimal* function in S; we claim this will be an absolute value.

Lemma 1. *For every positive integer n, $|x^n| = |x|^n$ for all $x \in L$.*

Proof. It will be enough to check that the function $x \mapsto |x^n|^{1/n}$ also belongs to S, as then the minimality of $|\cdot|$ will force the issue (because (c) ensures that $|x^n|^{1/n} \leq |x|$). Properties (a), (c), (d) are directly inherited from $|\cdot|$. As for (b), write

$$|(x+y)^n| \leq \sum_{i=0}^{n} \binom{n}{i} |x|^i |y|^{n-i} = (|x| + |y|)^n. \qquad \square$$

Lemma 2. *For all $x \in L^\times$, $|x||x^{-1}| = 1$.*

Proof. By (a) and (c), we have $|x||x^{-1}| \geq 1$, so it suffices to obtain a contradiction under the assumption that $|x||x^{-1}| > 1$. In this case, we can choose ρ with $|x^{-1}|^{-1} < \rho < |x|$. Let R be the ring of power series $\sum_{n=0}^{\infty} a_n T^n$ over K for which $|a_n|\rho^n \to 0$ as $n \to \infty$; since K is complete, R is a commutative Banach algebra over K for the submultiplicative norm

$$|\sum_n a_n T^n| \mapsto \sup_n \{|a_n|\rho^n\}.$$

Using the fact that the geometric series $1/(1-y)$ converges for $y \in R$ sufficiently small, we see that the set of units of R is open in R.

By expanding out $(1 - x^{-1}T)^{-1}$ and using Lemma 1, we see that $T - x$ is not a unit in R. Since the units in R are open, it follows that the closure of the ideal $T - x$ is not the unit ideal, so the quotient of R by this closure is a nonzero Banach ring for the quotient norm. (The quotient norm of a residue class is the infimum of the norms of its representatives.) If we restrict the quotient norm back to K, we get a function which again belongs to S and is bounded above by $|\cdot|$, but its value on x is at most ρ; this contradicts the minimality of $|\cdot|$. □

Lemma 3. *The function $|\cdot|$ is a multiplicative valuation extending K.*

Proof. The fact that $|\cdot|$ belongs to S implies everything except multiplicativity. For this, use Lemma 2 to write

$$|xy| \leq |x||y| \leq |x||x^{-1}||xy| = |xy|.$$

□