Dirichlet's units theorem

HW 3 is posted and assigned in Cocalc.

real complex

(r_1, r_2) = signature of K

K = # field

Thm $\mathcal{O}_K^\times$ is finitely generated abelian group

more precisely, $\mu(K)$ is finite and $\dfrac{\mathcal{O}_K^\times}{\mu(K)}$ is finite free of rank $r_1 + r_2 - 1$.

Reminder: the multiplicative Minkowski space

$$K^* \to K_{\mathbb{R}}^* \xrightarrow{\log} \mathbb{R}^{r_1+r_2}$$

$$(z_\tau)_\tau \longrightarrow (\log|z_\tau|)_\tau$$

units land in subspace $H: \sum_\tau x_\tau = 0$. of $\mathcal{O}_K^*$

want to show: $\dfrac{\mathcal{O}_K^*}{\mu(K)}$ is a lattice in $H \left(\cong \mathbb{R}^{r_1+r_2-1}\right)$

Last time: established discreteness

need to show: cocompactness.

The units in the Minkowski space

$$\begin{tikzcd}
\mathcal{O}_K^* / \mu(K) \arrow[r] \arrow[d, hook] & H \arrow[d, hook] \\
K^* / \mu(K) \arrow[r] & \mathbb{R}^{r_1 + r_2}
\end{tikzcd}$$

A lemma on lattices

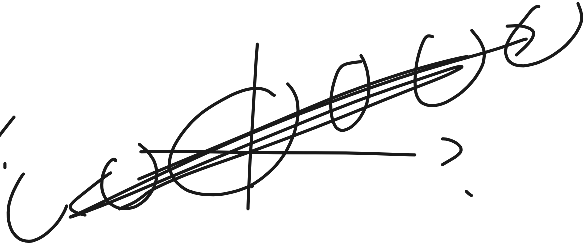
Lemma: Let $L \subset V$ be a discrete subgroup. (V — finite dim $\mathbb{R}$-vectorspace)

Then L is cocompact (i.e. a lattice)

<u>iff</u> $\exists$ a <u>bounded</u> subset M of V s.t.

$$V = \bigcup_{\alpha \in L} (M+\alpha) \qquad (M+\alpha = \{m+\alpha : m \in M\})$$

<u>Pf</u> This implies $\operatorname{rank} L = \dim V$.



Units from integers of small norm

Recall For any $N > 0$, there are only finitely many (principal) ideals of $\mathcal{O}_K$ of norm N.

So, given many elements of $\mathcal{O}_K$ of small absolute norm, get collisions: $\alpha_1 \neq \alpha_2 \in \mathcal{O}_K$ with $(\alpha_1) = (\alpha_2)$

$\alpha_1 \mathcal{O}_K = \alpha_2 \mathcal{O}_K$.

$\Rightarrow \alpha_2 = \alpha_1 \cdot u \quad \boxed{u \in \mathcal{O}_K^*}$

Casting the net for integers of small norm

Pick $c_\tau > 0$ s.t. $C = \prod_\tau c_\tau > \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$

$X = \{ (z_\tau) \in K_{\mathbb{R}} : |z_\tau| < c_\tau \}$

For $y \in K_{\mathbb{R}}^*$ $\prod_\tau |y_\tau| = 1$, (i.e. $\log y \in H$)

$X_y = \{ (z_\tau) \in K_{\mathbb{R}} : |z_\tau| < c_\tau' \}$ $\quad c_\tau' = c_\tau |y_\tau|$.

By Minkowski, $\exists\, \alpha \neq 0 \in \mathcal{O}_K \cap X_y$

In particular, $|\mathrm{Norm}_{K/\mathbb{Q}}(\alpha)| \leq C$

So α generates one of finitely many ideals, i.e.

can fix some finite system of elements $\alpha_1 \dots \alpha_m \in \mathcal{O}_K$

s.t. this α is associated to one of these.

Conclusion of the proof

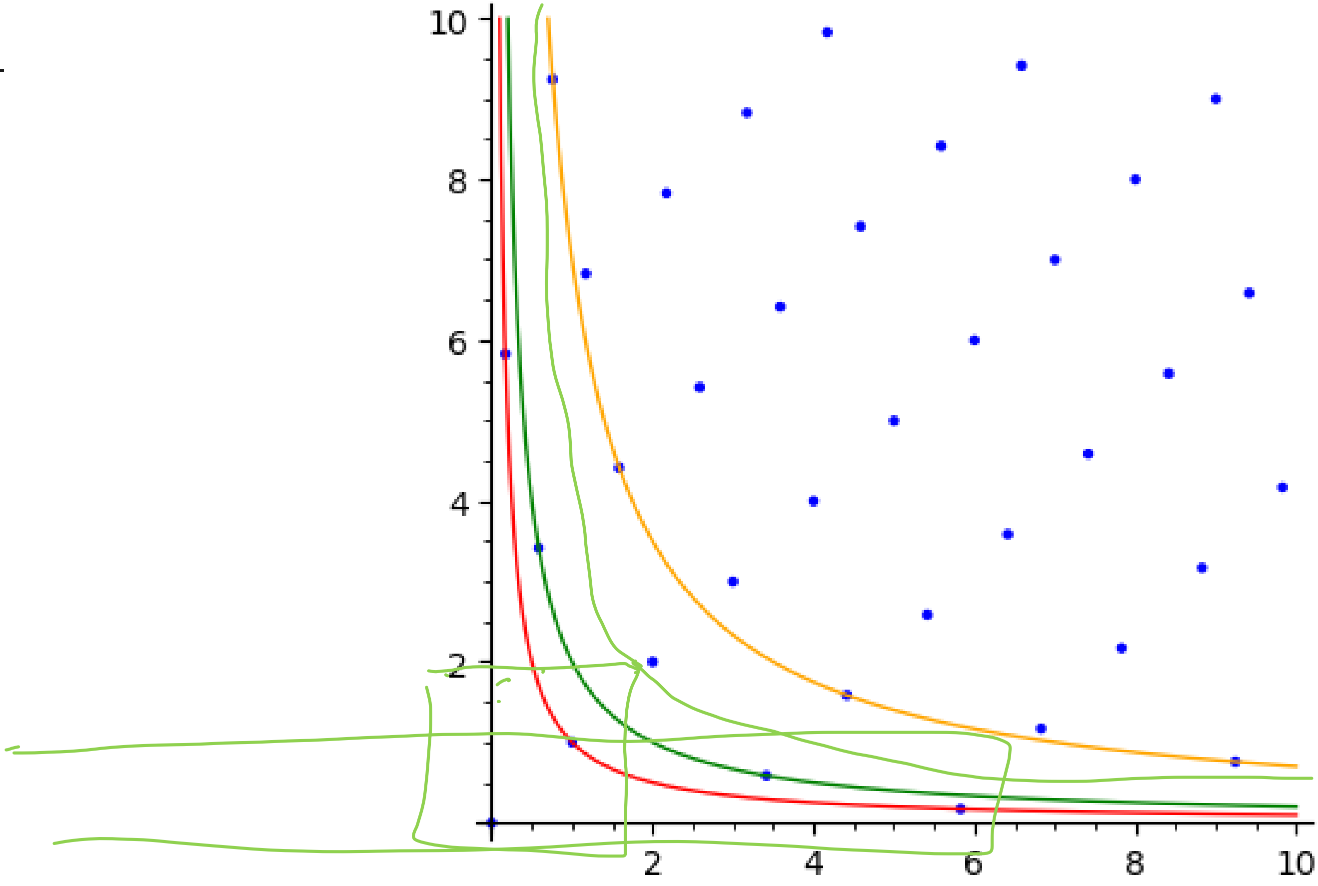
$\alpha_1 \ldots \alpha_m \in \mathcal{O}_K$

Consequently, $\log \left(\bigcup_{j=1}^{m} X \cdot \alpha_j^{-1} \right) \cap H$

has the property that its translates by $\mathcal{O}_K^\times / \mu(K)$ cover H.

$\Rightarrow \mathcal{O}_K^\times / \mu(K)$ is cocompact in H.

Illustration



The unit regulator

The <u>unit regulator</u> of K is the <u>covolume</u> of $\mathcal{O}_K^*/\mu(K)$ in $H \subset \mathbb{R}^{r_1+r_2}$

e.g. if $(r_1, r_2) = (2, 0)$ (real quadratic case)

~~look~~ let $\alpha \in \mathcal{O}_K^*$ be a fundamental unit and take $\sqrt{2} \cdot \left| \log |\alpha|_K \right|$

In general, take multiplicative generators $\alpha_1, \ldots, \alpha_{r_1+r_2-1}$ of $\mathcal{O}_K^*/\mu(K)$

and look at maximal minor of $r_1+r_2 \left\{ \underbrace{\left| \begin{matrix} \log \tau(\alpha_i) & \cdots & \cdot \\ \vdots & & \\ \vdots & & \end{matrix} \right|}_{r_1+r_2-1} \right.$

(times $\sqrt{r_1+r_2}$)

Context: Given K a # field, we have its <u>classnumber</u> $h_K = \# Cl_K$ & <u>unit regulator</u> R_K

$$\begin{pmatrix} c \\ -c \end{pmatrix}$$

$$\begin{array}{cc} & n \\ n+1 & \underbrace{\begin{pmatrix} \cdot & & \\ \vdots & & \\ \cdot & \cdots & \cdot \end{pmatrix}}_{M} \end{array}$$

$$\overbrace{(1 \; \cdots \; 1)} M = 0.$$

these can vary in complicated ways, but...

Dirichlet's class number formula

Gives an explicit (analytic) formula for

$$\boxed{h_K R_K}$$

$\mathbb{Q}(\sqrt{-D})$

e.g. if K is imaginary quadratic, then

$R_K = 1$, $h_K \to \infty$ as $D \to \infty$

$\mathbb{Q}(\sqrt{D})$

if K is real quadratic, then

R_K can be large, $h_K = \underline{1}$ apparently infinitely often