Localization

Wednesday, November 11 is a University holiday. On that day:

- No lecture or morning office hours.
- Evening office hours will meet as usual. 
  \[
  \{8-9\text{ pm} \quad \rho \leq T = \nu (C - 8)\}
  \]
Localization in the rational integers: $S$-units

$S$ is a finite set of prime numbers (e.g., $\{2, 3\}$)

$\mathbb{Z}_S = \{ \frac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z}, b \text{ only divisible by primes in } S \}$

$S$-integer

$s$-units in $\mathbb{Z}_S = \left\{ \frac{a}{b} \in \mathbb{Q} : a, b \not\equiv 0, \text{ any divisor by primes in } S \right\}$

$\mathbb{Z}_S^\times \subseteq \mathbb{Z}_S$
The S-unit theorem

The equation \( x + \frac{1}{x} = 1 \) has only finitely many solutions in S-units.

(May refer to this as a theorem)

(For many has a solver)

\[
\frac{3}{2} + \frac{-1}{2} = \frac{1}{2}
\]

\[
9 + (1 - 8) = 2
\]

\[
3 + (1 - 2) = 2
\]
Localization in the rational integers: at a single prime

\[ \mathbb{Z} = \mathbb{Z} \times \mathbb{Q} \]

\[ \mathbb{Z}_p = \left\langle \frac{a}{b} \in \mathbb{Q} ; \ a \in \mathbb{Z}, \ b \in \mathbb{Z}, b \neq 0, b \text{ not divisible by } p \right\rangle \]

is a ring

\[ (p) \mathbb{Z}_p \rightarrow \mathbb{F}_p \]

prime

In particular, \( \mathbb{Z}_p \) is a discrete valuation ring.
Localizations of integral domains

\[ A = \text{integral domain} \]
\[ K = \text{Fraction} \quad A = \left\langle \frac{a}{b} : a, b \in A, \ b \neq 0 \right\rangle / \sim \]

Note: if \( A \neq \mathbb{Z} \), then "inversion" may not make sense.

Let \( S \) be a subset of \( A - \{0\} \) closed under multiplication.

Define \( S^{-1} A = SA^{-1} = \left\{ \frac{a}{b} \in K : a \in A, \ b \in S \right\} \)

This is a ring and \( A \to SA \) is injective.

e.g. \( S^{-1}(1) = S^{-1}A = A, \) \( S = A - \{0\} \), \( S^{-1}A = K. \)
Localization at a prime ideal

$A =$ integral domain
$p =$ prime ideal
$n A = A - f \text{ is multiplicative!}$
$m A = S^{-1} A$

Proof: This is a localization
It is unique maximal ideal, namely
$L \subseteq A_p = \text{residue field is } \text{Frac}(A/p)$
For any positive integer $n$, $A/f^n = A_f/(fA_f)^n$
Discrete valuation rings

A DVR is a PID Ω with a unique maximal ideal \( \mathfrak{m} \neq 0 \).

(c.f. \( \mathbb{Z}[[p]] \)).

\[ \mathbb{P} \]

Proof: Since \( \mathfrak{m} \) is maximal, some ideals of \( \mathfrak{m} \) correspond to prime ideals of \( A \) not intersecting \( S \).

\[ \mathfrak{m} \rightarrow \mathfrak{m} A \]

If \( I \cap A = I \)

Char: every nonzero element of \( A \) is of form \( u \mathfrak{m} = u \mathfrak{m} \mathfrak{m} \mathfrak{m} \mathfrak{m} \cdots \).

What if \( x \notin \mathfrak{m} \)?

\[ \Rightarrow x = 0 \quad \text{or} \quad \mathfrak{m} \mathfrak{m} \mathfrak{m} \mathfrak{m} \cdots \in \mathfrak{m} \mathfrak{m} \mathfrak{m} \mathfrak{m} \cdots \]

\[ V: K^\times \rightarrow \mathbb{Z} \quad V(a) = \mathfrak{m} \mathfrak{m} \mathfrak{m} \mathfrak{m} \cdots \in \mathfrak{m} \mathfrak{m} \mathfrak{m} \mathfrak{m} \cdots \quad \text{s.t.} \quad a = u \mathfrak{m} \mathfrak{m} \mathfrak{m} \mathfrak{m} \cdots \in \mathfrak{m} \mathfrak{m} \mathfrak{m} \mathfrak{m} \cdots \]

\[ K = \mathbb{R} \quad \forall \sqrt{a/b} = V(a) - V(b) \]
Localizations of Dedekind domains

Proposition: If \( R \) is a Dedekind domain and \( S \subseteq R - \{0\} \) is a multiplicative subset, then \( S^{-1}R \) is also a Dedekind domain.

Proof: - Noetherian (\( \subseteq R \) properties)
- Every nonzero ideal is maximal (see previous slide)
- Internally closed in field of fractions

Given \( x \in K \), let \( s \) satisfy \( x^n + \frac{a_1}{s^{n-1}}x^{n-1} + \cdots + \frac{a_n}{s^n} = 0 \) with \( a_i \in R \) and \( s \in S \).

If \( x \in R \), then \( x/s \in S^{-1}R \).

Hence, \( R \) is located in \( S^{-1}R \).
Dedekind domains and discrete valuation rings

\[ R = \text{noetherian integral domain.} \]

The \( R \) is Dedekind domain.

\[ \langle \text{For every prime ideal } \mathfrak{p} \rangle \]

\[ R_{\mathfrak{p}} \text{ is a DVR.} \]

Forsw revenue \( \langle f \rangle \) dire chain.

In \( R_{\mathfrak{p}} \), ideals are powers of \( \mathfrak{p} \) plus \( 0 \) (by unique for \( \mathfrak{p} \), when of ideals in \( R \)).
Valuations on the rational integers

On \( \mathbb{Q} \), one can define \( \text{val}_p \) the

\[ p \text{-adic valuation of } \]

for any prime \( p \)

(e.g., by restriction from \( \mathbb{R} \))
Valuations on a Dedekind domain

For $R$ a Dedekind domain, for $a$ a prime ideal, define a $p$-adic valuation

$$V_p: R - \langle a \rangle \to \mathbb{Z}$$

$$V_p(f) - \text{exponent of } f \text{ in factorization of } (a)$$