Localization

Wednesday, November 11 is a University holiday. On that day:

- No lecture or morning office hours.
- Evening office hours will meet as usual. (8-9pm PST = UTC - 8)

Localization in the rational integers: S-units

S = finite set of prime numbers (e.g. $\{2,3\}$)

$\mathbb{Z}_S = \left\{ \frac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z}, b \neq 0 \text{ only divisible by primes in } S \right\}$ S-integers

is a ring $\mathbb{Z} \subset \mathbb{Z}_S \subset \mathbb{Q}$

$\underline{S\text{-units}}$ = units in $\mathbb{Z}_S = \left\{ \frac{a}{b} \in \mathbb{Q} \;\; a, b \neq 0, \text{ only divis. by primes in } S \right\}$

(e.g. $\pm 2^i 3^j \quad i, j \in \mathbb{Z}$)

$\frac{5}{7} \in \mathbb{Z}_S - \mathbb{Z}_S^\times$

The S-unit theorem

Ex S = {2, 3}

Theorem The equation $x + y = 1$ has only finitely many solutions in S-units.

(Many proofs of this, algorithms)

(SageMath has a solver)

$3 + (-2) = 1$

$9 + (-8) = 1$

$\frac{3}{2} + \frac{-1}{2} = 1$

Localization in the rational integers: at a single prime

p = prime

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} ; \begin{array}{l} a \in \mathbb{Z} \\ b \in \mathbb{Z} - \{0\} \end{array} \; b \text{ not divisible by } p \right\}$$

is again a ring

$$(p) \subset \mathbb{Z}_{(p)} \twoheadrightarrow \mathbb{F}_p$$

prime

only ideals in this ring are $(p), (p^2), (p^3) \ldots$ and (0)

prime

in particular, $\mathbb{Z}_{(p)}$ is a discrete valuation ring

Localizations of integral domains

A = integral domain

$K = \mathrm{Frac}\, A = \{ \frac{a}{b} : a, b \in A,\ b \neq 0 \} / \sim$

$\frac{a}{b} \sim \frac{c}{d} \iff ad = bc.$

Note: if A not a PID, the "lowest terms" may not make sense.

Let S be a subset of $A - \{0\}$ containing 1 and closed under multiplication.

Define $S^{-1}A = AS^{-1} := \{ \frac{a}{b} \in K : a \in A, b \in S \}$

This is a ring and $A \to S^{-1}A$ is injective

e.g. $S = \{1\}$, $S^{-1}A = A$; $S = A - \{0\}$, $S^{-1}A = K$.

Localization at a prime ideal

A = integral domain

$\mathfrak{p}$ = prime ideal

note: $S = A - \mathfrak{p}$ is multiplicative!

define $A_{\mathfrak{p}} = S^{-1}A$.

(e.g. $A = \mathbb{Z}$, $\mathfrak{p} = (p)$

$A_{\mathfrak{p}} = \mathbb{Z}_{(p)}$)

Prop this is a local ring

i.e. unique maximal ideal, namely

$\mathfrak{p}A_{\mathfrak{p}}$ (residue field is $\mathrm{Frac}(A/\mathfrak{p})$)

For any positive integer n, $A/\mathfrak{p}^n \cong A_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})^n$

Discrete valuation rings

def A DVR is a PID $\mathcal{O}$ with a unique maximal ideal $\mathfrak{p} \neq 0$.
$\left(\text{e.g. } \mathbb{Z}_{(p)}\right)$. (π)

Prop in general, prime ideals of $S^{-1}A$ correspond to prime ideals of A not intersecting S via

$$\mathfrak{q} \longrightarrow \mathfrak{q}S^{-1}A$$
$$I \cap A \longleftarrow I$$

Claim: every nonzero element of $\mathcal{O}$ is of form $u\pi^n$ $n \geq 0$
(what if $x \in \mathcal{O}$ is divisible by $\pi, \pi^2, \pi^3, \ldots$?) $u \in \mathcal{O}^*$.
$\Rightarrow x = 0$. $a \in \mathcal{O} - \{0\}$

$V: K^* \to \mathbb{Z}$ $\quad V(a) =$ integer n s.t. $a = u\pi^n$ with $u \in \mathcal{O}^*$

$K = \mathrm{Frac}\,\mathcal{O}$ $\quad V(a/b) = V(a) - V(b)$

Localizations of Dedekind domains

prop If R is a Dedekind domain
$S \subseteq R - \{0\}$ multiplicative subset
then $S^{-1}R$ is also a Dedekind domain

pf - noetherian ($\Leftarrow R$ noetherian)
- every nonzero prime ideal is maximal (see previous slide)
- integrally closed in field of fractions

$x \in K$ satisfies $x^n + \frac{a_1}{s_1} x^{n-1} + \dots + \frac{a_n}{s_n} = 0$ $\quad a_i \in R$, $s_i \in S$

$\operatorname{Frac} R = \operatorname{Frac} S^{-1}R$

$\Rightarrow$ for $s = s_1 \cdots s_n$, sx integral over R, hence in R.

Dedekind domains and discrete valuation rings

Prop R = noetherian integral domain. (e.g. $\mathbb{Z} = \mathcal{O}_K$, K = # field)

Then R is Dedekind domain

$\Longleftrightarrow$ For every prime ideal $\mathfrak{p} \neq 0$ $R_{\mathfrak{p}}$ is a DVR.

Focus now on "$\Leftarrow$" direction.

in $R_{\mathfrak{p}}$, ideals are powers of $\mathfrak{p}R_{\mathfrak{p}}$ plus 0 (by unique factorization of ideals in R)

Valuations on the rational integers

on $\mathbb{Z}$, I can define

p-adic valuation v_p

for any prime p ...

(e.g. by restriction from $\mathbb{Z}_{(p)}$)

Valuations on a Dedekind domain

For R a Dedekind domain, $\mathfrak{p} \subset R$ a prime ideal, define a $\mathfrak{p}$-adic **valuation**

$$\begin{tikzcd} v_{\mathfrak{p}} : R - \{0\} \arrow[rr] \arrow[dr] & & \mathbb{Z} \\ & (\mathrm{Frac}\, R)^{\times} \arrow[ur] & \end{tikzcd}$$

$v_{\mathfrak{p}}(\alpha) :=$ exponent of $\mathfrak{p}$ in factorization of (α)