More on localization

PS6 posted to web site; coming to CoCalc shortly.

If you are still working on PS5, check Zulip for several corrections (mea culpa).

Reminder: localization of an integral domain

A = integral domain

$K = \mathrm{Frac}(A) = \left\{ \frac{a}{b} : \begin{array}{l} a \in A, \\ b \in A - \{0\} \end{array} \right\} \Big/ \left(\frac{a}{b} = \frac{c}{d} \text{ if } ad = bc \right)$

S = $\underline{\text{multiplicative}}$ subset of $A - \{0\}$
containing 1

$S^{-1}A = \{ x \in K \text{ of form } \frac{a}{s}\ \ a \in A, s \in S \}$

e.g. $\mathfrak{p} \subset A$ prime ideal,
can take $S = A - \mathfrak{p}$

in this case
$S^{-1}A = A_{\mathfrak{p}}$ ← $\frac{\text{local ring}}{\text{max ideal } \mathfrak{p}A_{\mathfrak{p}}}$

Reminder: prime ideals in a localization

Prop Prime ideals of $S^{-1}A$ are all of form $\mathfrak{p}S^{-1}A$ where $\mathfrak{p} \subset A$ is a prime ideal not meeting S

(and this is a 1-1 correspondence).

Pf First observe that $\mathfrak{p}S^{-1}A = \left\{ \frac{q}{s} : \begin{array}{c} q \in \mathfrak{p} \\ s \in S \end{array} \right\}$

if $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{q}{s}$ $\quad q \in \mathfrak{p}$ $\quad s_1, s_2, s \in S \Rightarrow s \notin \mathfrak{p}$ $\quad s\underbrace{a_1}_{?} a_2 = q s_1 s_2 \in \mathfrak{p}$

$\mathfrak{p}$

if $\frac{q}{s} = \frac{a}{1} \Rightarrow a \in \mathfrak{p}$ $\qquad q \to q \cap A$ $\qquad \frac{a_1}{s_1} \in \mathfrak{p}S^{-1}A$

$\mathfrak{p} \longrightarrow \mathfrak{p}S^{-1}A \longrightarrow \mathfrak{p}$ $\quad \left(\begin{array}{l} q \subset S^{-1}A \\ (q \cap A) S^{-1}A = q \end{array} \right.$

Reminder: discrete valuation rings

A, $K = \mathrm{Frac}(A)$

A DVR is a PID with a unique maximal ideal $\mathfrak{p} \neq 0$.

(e.g. $\mathbb{Z}_{(p)}$, $K[[t]]$)

write $\mathfrak{p} = (\pi)$. Then ideals are of form

$(\pi), (\pi^2), \ldots$ and (0).

Get a function $v : K^\times \longrightarrow \mathbb{Z}$

characterized by: $v(x) = n$ iff $x\pi^{-n} \in \mathcal{O}_A^\times$

Key properties:

$$v(xy) = v(x) + v(y)$$

$$\boxed{v(x+y) \geq \min\{v(x), v(y)\}}$$

$3+3=6$

$3+6=9$

(with equality if $v(x) \neq v(y)$) "triangle inequality"

Localizations of Dedekind domains

<u>prop</u> Any localization of a Dedekind domain is still one.

<u>prop</u> Let $\mathcal{O}$ be a noetherian integral domain.
Then $\mathcal{O}$ is a Dedekind domain <u>iff</u> $\forall$ nonzero prime ideals $\mathfrak{p} \subset \mathcal{O}$, $\mathcal{O}_{\mathfrak{p}}$ is a DVR.

<u>Pf</u> $\Longrightarrow$ Suppose $\mathcal{O}$ is a Dedekind domain.
$\mathcal{O}_{\mathfrak{p}}$ is a Dedekind domain $\Rightarrow$ unique factorization of ideals
but only prime ideals of $\mathcal{O}_{\mathfrak{p}}$ are 0 and $\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$.

unique maximal ideal

Choose $\pi \in \mathfrak{p} - \mathfrak{p}^2$, then $\pi\mathcal{O}_{\mathfrak{p}} = \underline{\mathfrak{p}\mathcal{O}_{\mathfrak{p}}}$
Similarly, every nonzero ideal is $(\mathfrak{p}\mathcal{O}_{\mathfrak{p}})^n = (\pi)^n$ for some n.

Localizations of Dedekind domains (continued)

Suppose for $\mathfrak{p} \subset \mathcal{O}$ nonzero prime, $\mathcal{O}_\mathfrak{p}$ is a DVR.

Lemma: $\mathcal{O} = \bigcap \mathcal{O}_\mathfrak{p}$ inside $K = \operatorname{Frac} \mathcal{O}$.

true for any integral domain

Pf $\frac{a}{b} \in \bigcap \mathcal{O}_\mathfrak{p} \Rightarrow \{x \in \mathcal{O} : xa \in b\mathcal{O}\}$ is an ideal not contained in any $\mathfrak{p}$

$a, b \in \mathcal{O}$

$\left(\frac{a}{b} = \frac{c}{s},\ s \notin \mathfrak{p},\ sa = bc\right)$

$\Rightarrow 1 \in \ \Rightarrow \frac{a}{b} \in \mathcal{O}$.

Note: DVR = PID $\Rightarrow$ integrally closed (Gauss's lemma)

so each $\mathcal{O}_\mathfrak{p}$ is integrally closed,

hence $\bigcap \mathcal{O}_\mathfrak{p} = \mathcal{O}$ is too.

If $0 \neq \mathfrak{p} \subsetneq \mathfrak{q}$ then $\mathfrak{p}\mathcal{O}_\mathfrak{q}$ would be a nonzero prime of $\mathcal{O}_\mathfrak{q}$ which is a DVR $\Rightarrow\!\!\!\!\!\times$

Noetherian domain
integrally closed
every nonzero prime is maximal.

Valuations on the integers

$v : K^\times \to \mathbb{Z}$

$v(xy) = v(x)\, v(y)$

$v(x+y) \geq \min(v(x), v(y))$

$v(1) = 0$

For each positive prime integer $p \in \mathbb{Z}$

$v_p : \mathbb{Q} \setminus \{0\} \longrightarrow \mathbb{Z}$

$v_p^{-1}(\{0, 1, \dots\}) = \mathbb{Z}_{(p)}$ a DVR.

$v_p^{-1}(\{1, \dots\}) = p\mathbb{Z}_{(p)}$ maximal ideal.

Valuations on a Dedekind domain

$K = \text{frac } \mathcal{O}$

$\mathcal{O}$ = Dedekind domain. For each nonzero prime $\mathfrak{p} \subset \mathcal{O}$

can define $v_{\mathfrak{p}} : K^* \longrightarrow \mathbb{Z}$

where $v_{\mathfrak{p}}(\alpha)$ measures exponent of $\mathfrak{p}$ in factorization of $\alpha\mathcal{O}$ (as a fractional ideal)

again,

$$v_{\mathfrak{p}}^{-1}(\{0, 1, \ldots\}) = \mathcal{O}_{\mathfrak{p}}$$

$$v_{\mathfrak{p}}^{-1}(\{1, 2, \ldots\}) = \mathfrak{p}\mathcal{O}_{\mathfrak{p}}$$

Localization, class group, and rings of integers

Let $\mathcal{O}$ be a Dedekind domain, X = set of nonzero primes with <u>finite</u> complement.

$$\mathcal{O}(X) := \left\{ \frac{f}{g} : f, g \in \mathcal{O},\ g \not\equiv 0 \bmod \text{any } \mathfrak{p} \in X \right\}.$$

$$= S^{-1}\mathcal{O} \quad S = \{ s \in \mathcal{O} : s \not\equiv 0 \text{ mod any } \mathfrak{p} \in X \}.$$

<u>Prop</u> There is an exact sequence

$$1 \to \mathcal{O}^\times \to \mathcal{O}(X)^\times \to \bigoplus_{\mathfrak{p} \notin X} \underbrace{K^\times / \mathcal{O}_\mathfrak{p}^\times}_{\substack{\cong \mathbb{Z} \\ v_\mathfrak{p}}} \to Cl(\mathcal{O}) \to Cl(\mathcal{O}(X)) \to 1$$

$$(e_\mathfrak{p})_{\mathfrak{p} \notin X} \longmapsto \prod_{\mathfrak{p} \notin X} \mathfrak{p}^{e_\mathfrak{p}}$$

The S-unit group and the S-class group

K : # field

S = finite set of nonzero primes (X = complement of S)

$\mathcal{O}_K^S = \mathcal{O}_K(X) =$ S-integers

$(\mathcal{O}_K^S)^\times =$ S-units

real embeddings r_1 # complex pairs of embeddings r_2

Dir $(\mathcal{O}_K^S)^\times = \mu(K) \times \mathbb{Z}^{r_1 + r_2 - 1 + \#S}$

Cor $Cl(\mathcal{O}(X))$ is finite.

$= Cl_K^S$