Different and discriminant (part 1)

For enrolled students, HW 2 has been graded and feedback returned.

Vandermonde determinants and resultants

$$P(x) = (x-\alpha_1)\cdots(x-\alpha_n) \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{C}$$

$$V(\alpha_1, \ldots, \alpha_n) = \det \begin{vmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_n \\ \alpha_1^2 & \cdots & \alpha_n^2 \\ \vdots & & \\ \alpha_1^{n-1} & \cdots & \alpha_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i)$$

(comes from solving for a polynomial of degree $< n$ from values at $\alpha_1 \ldots \alpha_n$)

$$V(\alpha_1 \ldots \alpha_n)^2 = \pm \prod_{\substack{1 \le i,j \le n \\ i \ne j}} (\alpha_i - \alpha_j) = \prod_{1 \le i \le n} \prod_{j \ne i} (\alpha_i - \alpha_j)$$

Vandermonde determinants and resultants

$p(x) = (x-\alpha_1)\cdots(x-\alpha_n)$ $\quad p'(\alpha_i) = \prod_{j \neq i} \alpha_i - \alpha_j$

$$\boxed{\underline{V(\alpha_1 \ldots \alpha_n)^2} = \pm \prod_i p'(\alpha_i)}$$

$= \mathrm{Res}(p, p')$

resultant $\longrightarrow$

discriminant

L/K $\quad$ Disc $\leftarrow$ ideal of O_K

different $\leftarrow$ ideal of O_L

Trace pairings and dual modules

L/K extension of number fields

nondegenerate pairing

<u>trace pairing</u> $L \times L \to K$ $\langle x, y\rangle = \mathrm{Trace}_{L/K}(xy)$

where $\mathrm{Trace}_{L/K}(z)$ = trace of multiplication-by-z as a K-linear map on L

induces an isom as K-vector space $L \longrightarrow \mathrm{Hom}_K(L, K)$ dual vector space

$x \longrightarrow (y \mapsto \langle x, y\rangle)$

$I \subseteq L$ <u>fractional ideal</u>

<u>dual module</u> of $I = \underline{\{x \in L : \mathrm{Trace}_{L/K}(x \cdot I) \subseteq \mathcal{O}_L\}}$ ${}^{*}I$

is again a <u>fractional ideal</u>

$\boxed{{}^{*}({}^{*}I) = I}$

$\supseteq$ ✓ $\subseteq$??

The (inverse) different

<u>Inverse different</u> of L/K is $\overset{\text{fractional}}{\text{ideal}}$ $* \mathcal{O}_L$.

<u>Different</u> of L/K is $\overline{\text{integral}}$ ideal $(*\mathcal{O}_L)^{-1} \subseteq \mathcal{O}_L$

e.g. $K = \mathbb{Q}$, $L = \mathbb{Q}[i]$ $\quad Tr(1) = 2 \quad Tr(i) = 0$

inverse different is $(1+i)^{-1}\mathbb{Z}[i]$ $\quad$ Different is $(1+i)$

$$Tr((1+i)^{-1}) = 1 \qquad Tr(i(1+i)^{-1}) = 1.$$

$$= \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} \qquad \underbrace{(1+i)^{-1}}_{= \frac{1+i}{2}}$$

Properties of the different

$\mathfrak{D}_{L/K}$ = different of L/K $\subseteq \mathcal{O}_L$

~~prop.~~ For $K \subseteq L \subseteq M$,

$$\mathfrak{D}_{M/K} = \mathfrak{D}_{M/L}\underbrace{\mathfrak{D}_{L/K}}_{\text{extend ideal from } \mathcal{O}_L \text{ to } \mathcal{O}_M}$$

Define similar different for an extension of Dedekind domains (coming from finite separable extension of fields)

$\mathfrak{D}_{\mathcal{O}_L/\mathcal{O}_K}$

Then for any multiplicative set S of $\mathcal{O}_K$

$$\mathfrak{D}_{S^{-1}\mathcal{O}_L/S^{-1}\mathcal{O}_K} = S^{-1}\mathfrak{D}_{\mathcal{O}_L/\mathcal{O}_K}$$

Properties of the different

The different in the monogenic case

($\mathcal{O}_L$ is monogenic over $\mathcal{O}_K$)

Say $\mathcal{O}_L = \mathcal{O}_K[\alpha]/(P(\alpha))$

<u>Thm</u> $\mathfrak{D}_{\mathcal{O}_L/\mathcal{O}_K} = (P'(\alpha))$.

<u>Pf</u> $P(x) = a_0 + \cdots + a_n x^n \qquad a_i \in \mathcal{O}_K$

$$\frac{P(x)}{x-\alpha} = b_0 + \cdots + b_{n-1} x^{n-1} \qquad b_i \in \mathcal{O}_L$$

dual basis of $1, \alpha, \ldots \alpha^{n-1}$ wrt trace pairing is:

$$\frac{b_0}{P'(\alpha)}, \ldots\ldots, \frac{b_{n-1}}{P'(\alpha)}.$$

(b/c $\displaystyle\sum_i \frac{P(x)}{x-\alpha_i} \frac{\alpha_i^r}{P'(\alpha_i)} = x^r$ for $r \leq n-1$

The different in the monogenic case

$$\Rightarrow \mathrm{Trace}_{L/K}\left(\frac{f(x)}{x-\alpha}\frac{\alpha^r}{f'(\alpha)}\right) = x^r\Big)$$

$$\Rightarrow \text{inverse different is } f'(\alpha)^{-1}\underbrace{(b_0, \dots, b_{n-1})}_{\text{this is unit, ideal}}$$

fractional

$$(b_0 + b_1 x \cdots + b_{n-1}x^{n-1})(x-\alpha) = (a_0 + \dots + a_{n-1}x^{n-1} + x^n)$$

$$\Rightarrow b_{n-1} = 1$$

$$b_{n-2} - \alpha b_{n-1} = a_{n-1},$$

$$\vdots$$

$$\Rightarrow b_i \in \mathcal{O}_L,\ b_{n-1} = 1.$$

Preview: the different in the non-monogenic case

Theorem $\mathfrak{D}_{\mathcal{O}_L/\mathcal{O}_K}$ = ideal generated by $p'(\alpha)$ where α runs over all elements of $\mathcal{O}_L$ s.t. $L = K[\alpha]/(p(\alpha))$

Pf one way:

$\supseteq$ Use similar calculation to show that

$$(p'(\alpha)) = I\,\mathfrak{D}_{\mathcal{O}_L/\mathcal{O}_K}$$

$$I = \{x \in L : x\mathcal{O}_L \subseteq \mathcal{O}_K[\alpha]\}$$

How to go the other way?? ..

Easy if $\mathcal{O}_L$ is K-locally monogenic.

Different and discriminant

Theorem

$\mathrm{disc}\, \mathcal{O}_L/\mathcal{O}_K =$

$\mathrm{Norm}_{L/K}\, \mathcal{D}_{\mathcal{O}_L/\mathcal{O}_K}$

= ideal of $\mathcal{O}_K$ generated by

$\mathrm{Norm}_{L/K}\, x \quad \forall x \in \mathcal{D}_{\mathcal{O}_L/\mathcal{O}_K}$.

$\mathrm{disc}\, \mathcal{O}_L/\mathcal{O}_K \subseteq \mathcal{O}_K$

= ideal generated by

$\mathrm{disc}(\alpha_1 \dots \alpha_n)$

where $\alpha_1 \dots \alpha_n$ run over bases of L/K consisting of elements of $\mathcal{O}_L$

Want to show: $\mathfrak{P} \subseteq \mathcal{O}_K$.

$\mathfrak{p}$ ramified in $\mathcal{O}_L \iff \mathfrak{p}$ divides $\mathrm{disc}_{\mathcal{O}_L/\mathcal{O}_K} \iff \mathfrak{p}^{\mathcal{O}_L}$ not coprime to $\mathcal{D}_{\mathcal{O}_L/\mathcal{O}_K}$.