Different and discriminant (part 2)

From now on, I will try to start the post-lecture office hours as soon as possible after I end the lecture, rather than waiting until 11:00. They will still run until 11:30 as scheduled.
Reminder: different of a field extension

$L/K$ be an extension of number fields.

Inverse different of $L/K = $ maximal ideal

\[
\{ x \in L : \text{Tr}(x \cdot \mathcal{O}_L) \subseteq \mathcal{O}_K \}.
\]

different $\mathfrak{D} \subseteq \mathcal{O}_K$ is inverse of this maximal ideal.

(fixed primes all have an isomorphism $L \rightarrow \text{Hom}_{K}(\mathcal{O}_L, \mathcal{O}_K)$)

but not $\mathcal{O}_L \rightarrow \text{Hom}_{K}(\mathcal{O}_L, \mathcal{O}_K)$

\[
\mathfrak{p}_L : \mathcal{O}_L = \mathcal{O}_K(\mathfrak{p})/(\mathfrak{p}(\mathfrak{p}))
\]

if $\mathfrak{p}_L = \mathcal{O}_K(\mathfrak{p})/(\mathfrak{p}(\mathfrak{p}))$ then $\mathfrak{D} \cap \mathcal{O}_K = (\mathfrak{p}_L)$.
Reminder: different vs discriminant

\[
\text{Domain is a subset of } \mathbb{C}.
\]

\[
\text{Disc. of } \mathbb{C} \text{ is a subset of } \text{OK.}
\]

\[
\text{Prop. Disc. of } \mathbb{C} = \text{Norm. of } \text{OK.}
\]

\[
\text{for } \mathbb{K} \text{ in } \text{OK, the norm holds}
\]

\[
\text{then } \text{Domain} = \text{Domain of } \text{OK.}
\]

\[
\Rightarrow \text{Disc. of } \mathbb{C} = \text{Disc. of } \text{OK.}
\]

\[
\text{Em: } 2] \text{ Norm. of } \text{Disc. of } \text{OK.}
\]
Étale algebras

Let $K$ be a field.

An étale algebra over $K$ is a ring which is a finite direct sum of finite separable field extensions of $K$.

E.g. $\mathbb{Q} \bigoplus \mathbb{Q}(\sqrt{3})$ is a étale algebra over $\mathbb{Q}$ of degree $2 + 2 + 1 = 5$.

E.g. over $\mathbb{R}$, an étale algebra is a direct sum of copies of $\mathbb{R}$ and $\mathbb{C}$.

For $K$ a field (or étale algebra) over $\mathbb{Q}$,

$$K \hookrightarrow K_1 \mathbb{R} \hookrightarrow K_1 \mathbb{C}$$

with:

- $K_1 \mathbb{R} = K \otimes \mathbb{R}$
- $K_1 \mathbb{C} = K \otimes \mathbb{C}$
Etale algebras and the trace pairing

For $R$ an etale algebra over $K$, we have the pairing $\langle x, y \rangle = \text{Tr}_{12/K}(xy)$. This is again a nondegenerate perfect pairing:

$$R \rightarrow \text{Hom}_K(R, K), \quad x \mapsto (y \mapsto \langle x, y \rangle)$$

is a isomorphism.
Reduction modulo a prime

Let $L/K$ be a finite separable extension, $\mathcal{O}_K$ be a Dedekind domain with fraction field $K$, $\mathcal{O}_L$ be its integral closure of $\mathcal{O}_K$ in $L$. Let $f | \mathcal{O}_K$ be a prime factor with all $e_i = 1$.

For $p = 2, \ldots, r$ (and $Q_i/p_i$ is separable),

Then $\mathcal{O}_L/\mathfrak{p}_i \cong \mathbb{Z}[\mathfrak{p}_i]$. The primes on $\mathcal{O}_L/K$, $\mathcal{O}_L/\mathcal{O}_K$, and $\mathcal{O}_L/p_i \cong \mathbb{Z}/p_i \mathbb{Z}$ are distinct primes on $\mathcal{O}_L/K$. $\mathcal{O}_L/p_i \cong \mathbb{Z}/p_i \mathbb{Z}$ is a finite algebra over $\mathcal{O}_K/p_i \cong \mathbb{Z}/p_i \mathbb{Z}$.
Ramification, different, and discriminant

Let $L/K$ be an extension of fields.

Then $f$ is ramified in $L(=) K$ if $\text{Disc} \omega/K$ is not congruent to 0 in $\mathcal{O}_K$.

We already know:
if $f$ is ramified then $\mathcal{O}_L/f \mathcal{O}_K$.

Need: if $f$ is unramified then $f + \mathcal{O}_K$ or $f / \mathcal{O}_K$ to $\mathcal{O}_L$.
Ramification, different, and discriminant

If $f$ is ramified then $\mathfrak{p} \mathfrak{d}_L = \mathfrak{P}_1^e \mathfrak{P}_2^f \cdots \mathfrak{P}_r^g$, and $\mathfrak{d}_L / \mathfrak{d}_L \mathfrak{f}_L \subseteq \mathfrak{O}_L / \mathfrak{q}_i$ is a finite algebra over $\mathfrak{O}_K$.

So the primes must be perfect, that is,

$\mathfrak{d}_L \rightarrow \text{Hom}_K(\mathfrak{O}_L, \mathfrak{d}_L)$ becomes an isomorphism.

$\Rightarrow$ if come to $D_{\mathfrak{p}}x_i$.

(Choose a basis $\mathfrak{B}$ of elements of $\mathfrak{d}_L$ which reduce mod $\mathfrak{p}$ to a basis of $\mathfrak{O}_L / \mathfrak{q}_i$ and choose elements not divisible by $\mathfrak{p}$)
Let $L/K$ be an extension of fields.

Let $f$ be a prime, let $g$ of $L$ be an integer.

- $e(\mathfrak{f}/\mathfrak{p}) = 1$ if $\forall g \in (\mathfrak{D}_{\text{wild}})$

- $v_g(\mathfrak{D}_{\text{wild}}) \geq e(\mathfrak{f}/\mathfrak{p}) - 1$ (the wild case)

- $v_g(\mathfrak{D}_{\text{wild}}) \leq e(\mathfrak{f}/\mathfrak{p}) - 1 + v_g(e(\mathfrak{f}/\mathfrak{p}))$

Note: $v_g(\mathfrak{D}_{\text{wild}})$ as a function of $g$.
The Hermite-Minkowski theorem

\[ \text{Theorem (Hermite-Minkowski)} \]

Let \( \mathbb{K} \) be a field, let \( S \) be a finite set of (concrete) pairs of \( \mathbb{K} \).

Then there are finitely many number fields \( \mathbb{L}/\mathbb{K} \) of (relative) degree \( n \) which are unramified at \( S \). 

i.e., unramified extensions of \( \mathbb{K} \)
The Hermite-Minkowski theorem

We assume $K = \mathbb{R}$.

Consider:

- Bounding $S$ also gives a bound on

\[ \text{Disc} \cdot K \cdot \mathfrak{a} \]

- There are only finitely many $\mathfrak{a}$ of fixed degree

and fixed discriminant.

(Geometry of numbers)

Choose a "small" element of $O_L$ in $\mathfrak{a}$

Primitive

Let defined by a "small" polynomial