The structure of ramification groups

No assignment for next Thursday due to the Thanksgiving holiday.
Also, no lecture on Friday, November 27. (Lectures and office hours earlier in the week will be held as usual.)

PS7 will be posted sometime next week. It will be due

Decomposition and inertia groups

L/K be a Galois extension of number fields $G = \mathrm{Gal}(L/K)$

Let $\mathfrak{p} \subset \mathcal{O}_K$ be a nonzero prime ideal

Let $\mathfrak{q} \subset \mathcal{O}_L$ be a prime above $\mathfrak{p}$.

$G_{\mathfrak{q}}$ = decomposition group $= \{\sigma \in G : \sigma(\mathfrak{q}) = \mathfrak{q}\}$

$\cup\!\!\mid$

$I_{\mathfrak{q}}$ = inertia group $= \ker\Big(G_{\mathfrak{q}} \twoheadrightarrow \underbrace{\mathrm{Gal}\big(\mathcal{O}_L/\mathfrak{q} \,\big|\, \mathcal{O}_K/\mathfrak{p}\big)}_{\text{cyclic}}\Big)$

$\# I_{\mathfrak{q}} = e(\mathfrak{q}/\mathfrak{p})$ $\quad$ $\# G_{\mathfrak{q}}/I_{\mathfrak{q}} = f(\mathfrak{q}/\mathfrak{p})$

A filtration on the decomposition group

$s \in \mathbb{R}$

$$G_{\mathfrak{q},s} = \left\{ \sigma \in G_{\mathfrak{q}} : \begin{array}{l} \text{for all } \alpha \in \mathcal{O}_L \\ v_{\mathfrak{q}}(\sigma(\alpha) - \alpha) \geq s+1 \end{array} \right\}.$$

i.e. σ acts trivially on $\mathcal{O}_L/\mathfrak{q}^n$ for all $n \leq s+1$

$\Rightarrow G_{\mathfrak{q},s}$ is a subgrp.

$G_{\mathfrak{q},-1} = G_{\mathfrak{q}}$

$\cap$

$G_{\mathfrak{q},0} = I_{\mathfrak{q}}$

$\cap$

$G_{\mathfrak{q},1} \subseteq G_{\mathfrak{q},2} \subseteq \dots$

The filtration is exhaustive

For $s >> 0$, $G_{\mathfrak{q}, s} = \{e\}$.

(for each $\sigma \in G_{\mathfrak{q}} - \{e\}$, $\exists$ some s for which

$\sigma \notin G_{\mathfrak{q}, s}$.

otherwise σ would fix $\mathcal{O}_L / \mathfrak{q}^n$ for all n

but $\bigcap_n \mathfrak{q}^n = \{0\}$.

$G_{\mathfrak{q}} = G_{\mathfrak{q}, -1}$

$\cup\!\!\mid\; I_{\mathfrak{q}} = G_{\mathfrak{q}, 0} \supseteq G_{\mathfrak{q}, 1} \supseteq G_{\mathfrak{q}, 2} \cdots \quad \cdots = \{e\}$

The ramification filtration and the different

decomposition field

Suppose $G = G_{\mathfrak{q}}$ (otherwise, replace K by $Z_{\mathfrak{q}}$)

and $\mathcal{O}_{L,\mathfrak{q}} = \mathcal{O}_{K,\mathfrak{p}}(\alpha)$ for some α where $P(\alpha) = 0$ min poly

(always true; prove it later)

Then $\mathcal{D}_{\mathcal{O}_{L,\mathfrak{q}}/\mathcal{O}_{K,\mathfrak{p}}} = (P'(\alpha))$ let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ be conjugates of α in L

$$= \left(\prod_{i=2}^{n} (\alpha_1 - \alpha_i) \right)$$

$$= \left(\prod_{\sigma \in G - \{e\}} (\alpha - \sigma(\alpha)) \right)$$

$$= \mathfrak{q}^m \quad \boxed{m = \sum_{\sigma \in G - \{e\}} \min\{s : \sigma \notin G_{\mathfrak{q},s}\}}$$

The first quotient: the residue field action

$$G_{\mathfrak{q}} = G_{\mathfrak{q},-1} \trianglerighteq \underbrace{G_{\mathfrak{q},0}}_{= I_{\mathfrak{q}}} \trianglerighteq G_{\mathfrak{q},1} \trianglerighteq \cdots$$

note $G_{\mathfrak{q},s} \trianglelefteq G_{\mathfrak{q}}$: is normal.

$G_{\mathfrak{q},s} / G_{\mathfrak{q},s+1}$

$$G_{\mathfrak{q}} / I_{\mathfrak{q}} \cong \mathrm{Gal}\left(\mathcal{O}_L/\mathfrak{q} \;/\; \mathcal{O}_K/\mathfrak{p} \right) \cong \text{cyclic.}$$

The next quotient: action on a uniformizer

uniformizing parameter

$G_{\mathfrak{q},0}/G_{\mathfrak{q},1}$

$= I_{\mathfrak{q}}$

cyclic

Pick $\pi \in \mathfrak{q} - \mathfrak{q}^2$

$\pi \mathcal{O}_L = \mathfrak{q}^1 \dots$

$$\boxed{\mathbb{F}_{\mathfrak{q}} := \mathcal{O}_L/\mathfrak{q}}$$

For $\sigma \in I_{\mathfrak{q}}$, $\sigma(\pi) \equiv c\pi \mod \pi^2$ for some $c \in \mathcal{O}_L$

$\sigma \in G_{\mathfrak{q},1} \iff c \equiv 1 \mod \pi$

$\Longrightarrow$ immediate

$\Longleftarrow$ $\alpha \in \mathcal{O}_L$ $\sigma(\alpha) = \alpha$ for all $\alpha \in \mathcal{O}_L$

?

$\leftarrow G_{\mathfrak{q},1} = \{\sigma \in G_{\mathfrak{q}} : \frac{\sigma(\alpha)}{\alpha} \equiv 1 \mod \mathfrak{q}\}$

(Skip this for now)

$G_{\mathfrak{q},1} =$ cyclic $\cancel{\times}$

$\ker(I_{\mathfrak{q}} \to \mathbb{F}_{\mathfrak{q}}$

$\sigma \longmapsto c$

The higher quotients

$$(\sigma_1 \circ \sigma_2)(\pi) - \pi \equiv \underbrace{\sigma_1\sigma_2(\pi) - \sigma_2(\pi)}_{\sigma(c_2\pi + \cdots) - (c_2\pi + \cdots)} + \underbrace{\sigma_2(\pi) - \pi}_{\equiv c_2 \pi \bmod \pi^2}$$

$$\sigma_1$$

$$\equiv \sigma_1(c_2)\sigma_1(\pi) - \sigma_1(c_2)\pi + \sigma_1(c_2)\pi \mod \pi^2$$

$$\equiv \underbrace{\sigma_1(c_2)}_{\equiv c_2 \bmod \pi}(c_1\pi) \mod \pi^2$$

$$\equiv c_1 c_2 \pi \mod \pi$$

$$\frac{\sigma_1\sigma_2(\pi)}{\pi} \mod \pi$$

$$= \frac{\sigma_1\sigma_2(\pi)}{\sigma_2(\pi)} \cdot \frac{\sigma_2(\pi)}{\pi} \mod \pi \quad \checkmark$$

$$\equiv c_1 c_2 \mod \pi$$

(definition of c_1 <u>doesn't depend</u> on choice of π)

Conclusion: decomposition groups are [illegible]

$G_{\mathfrak{q},1}/G_{\mathfrak{q},2}$ Given $\sigma \in G_{\mathfrak{q},1}$

$$\frac{\sigma(\pi)}{\pi} \in \{\alpha \in \mathcal{O}_L : \alpha \equiv 1 \mod \mathfrak{q}\} = U_{L,\mathfrak{q},1}$$

$$\sigma \in G_{\mathfrak{q},2} \iff \frac{\sigma(\pi)}{\pi} \in \{\alpha \in \mathcal{O}_L \mid \alpha \equiv 1 \mod \mathfrak{q}^2\} =: U_{L,\mathfrak{q},2}$$

abelian

$$\frac{G_{\mathfrak{q},1}}{G_{\mathfrak{q},2}} \hookrightarrow \frac{U_{L,\mathfrak{q},1}}{U_{L,\mathfrak{q},2}} \simeq \mathbb{F}_{\mathfrak{q}}$$ (additive group) ($\Leftarrow$ really one-dim vec space over $\mathbb{F}_{\mathfrak{q}}$)

$$(1+\pi\beta)(1+\pi\gamma) \equiv 1+\pi(\beta+\gamma) \mod \pi^2$$

Inverse limits

Could have done all of this with

$\mathcal{O}_L$ replaced by $\underline{q\text{-adic completion}}$

inverse limit $= \varprojlim_n \mathcal{O}_L/q^n$

$(\bar{\alpha}_1, \bar{\alpha}_2 \ldots\ldots)$ $\quad \bar{\alpha}_n \in \mathcal{O}_L/q^n$

$\alpha_{n+1} \bmod q^n = \alpha_n.$