The p-adic absolute value

Reminder: no lecture Friday. Happy Thanksgiving!

HW 7 posted, due Thursday, December 3.

The website for Math 204B is live, but it doesn't say much yet. There will continue to be an epicourse (using the same Zulip).

I just learned about a new "class number one" theorem for Galois CM fields! See this lecture's thread in Zulip.

Metric spaces

X = set

$d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$

d is a <u>metric</u> if: $\forall x_1, x_2 \in X$:

- (identity) $d(x_1, x_2) = 0 \iff x_1 = x_2$
- (symmetry) $d(x_1, x_2) = d(x_2, x_1)$
- (triangle inequality) $\forall x_3 \in X$

$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$

Cauchy sequences

(X, d) = metric space

A sequence $x_1, x_2 \ldots$ is <u>Cauchy</u> if

$\forall \varepsilon > 0, \exists N \geq 0$ s.t. for $i, j \geq N$, $d(x_i, x_j) < \varepsilon$.

A sequence $x_1, x_2 \ldots$ <u>converges to $x \in X$</u> if

$\forall \varepsilon > 0 \ \exists N > 0$ s.t. $\forall i \geq N \quad d(x_i, x) < \varepsilon$.

Notes:
- Any convergent sequence is Cauchy.
- Limit is unique if it exists.
- both properties are invariant under rearrangement.

The completion of a metric space

(X, d) = metric space

The completion $\hat{X}$ of X is set of equivalence classes of Cauchy sequences in X under relation $(x_1, x_2, \dots) \sim (y_1, y_2, \dots)$ if $(x_1, y_1, x_2, y_2, \dots)$ is a Cauchy seq.

This is idempotent.

$d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \lim_{N \to \infty} \sup\{ d(x_i, y_j) : i, j \geq N \}$

$\hat{X}$ is complete (every Cauchy sequence has a limit)

and X is complete $\iff$ the map $X \hookrightarrow \hat{X}$, $x \mapsto (x, \dots)$ is a bijection.

The real numbers as a completion

$X = \mathbb{Q}$, $d(x, y) = |x - y|_\infty$ ← usual absolute value

$\hat{X} = \mathbb{R}$ since $+$, $-$, $\times$ are continuous on $\mathbb{Q}$ they extend to $\mathbb{R}$.

So does $\div$

(if $x_1, x_2 \ldots$ a Cauchy sequence in $\mathbb{Q}$ not converging to 0, then $\exists \varepsilon > 0$ $|x_i| > \varepsilon$ $\forall i \gg 0$

uses $|xy|_\infty = |x|_\infty |y|_\infty$

$\Rightarrow$ $\mathbb{R}$ is a field.

The p-adic metric

Fix a prime p

$X = \mathbb{Q}$, $\quad d(x,y) = |x-y|_p$

where $|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-v_p(x)} & \text{if } x \neq 0 \end{cases}$

Can take metric completion to get

$\mathbb{Q}_p^{\text{metric}}$ — again a field!

$v_p(x,z) \geq \min(v_p(x,y), v_p(y,z))$

$\Rightarrow$

$|x-z|_p \leq \max\{|x-y|_p, |y-z|_p\}$

Strong triangle inequality

Note: Cauchy sequences converging to 0 form an ideal in ring of Cauchy sequences.

The product formula

On $\mathbb{R}$, I have defined $|\ |_\infty$ and $|\ |_p$ for every prime p.

archimedean.

Why is $|x|_p = p^{-v_p(x)}$?

Why p? Could have used any real number > 1

This choice means that for $x \neq 0$,

$$\prod_{v \in \{\text{primes}\} \cup \{\infty\}} |x|_v = 1$$

Pf: both sides are multiplicative and equal when $x = -1$ or $x = p$.

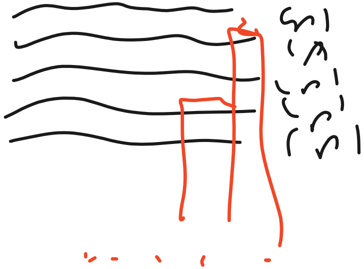
analogous to: a meromorphic function on a Riemann surface has total degree 0.

The p-adic numbers as a metric completion

Let $\mathbb{Q}_p = \mathbb{Z}_p[p^{-1}]$ where $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

- The p-adic valuation & absolute value <u>both</u> extend from $\mathbb{Q}$ to $\mathbb{Q}_p$.
- For the induced metric on $\mathbb{Q}_p$, $\mathbb{Q}$ is <u>dense</u> (in fact $\mathbb{Z}[p^{-1}]$ is also dense)
- $\mathbb{Q}_p$ is complete!

$$\Rightarrow \mathbb{Q}_p = \mathbb{Q}_p^{\text{metric}}$$



Convergent power series

Say $\sum a_n x^n \in \mathbb{Q}_p[[x]]$. Then for a given value $t \in \mathbb{Q}_p$,

can evaluate $\sum a_n t^n$ provided that

$$\lim_{n \to \infty} a_n t^n = 0 \iff \lim_{n \to \infty} |a_n|_p |t|_p^n$$

e.g. if $a_n \in \mathbb{Z}_p$, $x \in p\mathbb{Z}_p$, get convergence

(e.g. geometric series)

$$\text{e.g. } (1+x)^{1/n} = \sum_{i=0} \binom{1/n}{i} x^i \qquad \binom{1/n}{i} = \frac{(1/n)(1/n - 1)\cdots(1/n - i + 1)}{i!}$$

Another description of the p-adic integers

Cor (homework)

$$\mathbb{Z}_p \cong \mathbb{Z}[[x]]/(x-p)$$

what $\mathbb{Z}((x)) = \mathbb{Z}[[x]][x^{-1}]$

Idea:

$$\begin{tikzcd}
\mathbb{Z}[[x]] \arrow[r, "\text{evaluate at } x=p"] \arrow[dr, dashed] & \mathbb{Z}_p \\
& \mathbb{Z}[[x]]/(x-p) \arrow[u, "\cong ?"']
\end{tikzcd}$$

Cor

$$\mathbb{Q}_p \cong \mathbb{Z}((x))/(x-p)$$

Next time:

study generalization to completions any number field K with respect to $\mathfrak{p}$-adic valuation for any nonzero prime $\mathfrak{p}$ of $\mathcal{O}_K$.

$\underline{Q}$ How does this relate to $\mathbb{Q}_p$ where p is prime of $\mathbb{Q}$ below $\mathfrak{p}$?