Valuations

Normal schedule this week and next! (Last lecture December 11.)

PS 7 has been posted (but see corrections on Zulip).

Enrolled UCSD students can submit course evaluations until December 14. Please do!

The Math 204B home page is up; that course begins Monday, January 4. The course will be based in part on my notes on class field theory.

Multiplicative valuations (absolute values)

Let K be a field. A multiplicative valuation (absolute value) on K is a function $|\cdot|: K \longrightarrow \mathbb{R}$ s.t.

1) $|x| \geq 0$, $|x| = 0 \iff x = 0$ $\forall x \in K$

2) $|xy| = |x| \cdot |y|$ $\forall x, y \in K$

3) $|x+y| \leq |x| + |y|$ ($\Delta\leq$) $\forall x, y \in K$
triangle inequality

The trivial valuation is $|x|_{triv} = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$.
usually exclude this case.

Equivalence of valuations

For $|\,|: K \to \mathbb{R}$ a absolute value, (K, d) is a metric space where $d(x,y) = |x-y|$.

Get induced metric topology

Two absolute values $|\cdot|_1, |\cdot|_2$ are equivalent if they define same metric topology.

e.g. this occurs if $|\cdot|_2 = |\cdot|_1^c$ for some $c>0$. (same open balls)

prop Conversely, if $|\cdot|_1$ & $|\cdot|_2$ are equivalent, then $\exists\, c>0$ s.t. $|x|_2 = |x|_1^c$ for all $x \in K$.

A criterion for equivalence of valuations

Suppose $|\cdot|_1$, $|\cdot|_2$ are equivalent (and both nontrivial)

Then for $x \in K$

$$\underbrace{|x|_1 < 1}_{} \iff \{1, x, x^2, \dots\} \text{ converges to } 0 \iff \underbrace{|x|_2 < 1}_{}$$

x ~~is topologically nilpotent~~

Pick $y \in K$ s.t. $|y|_1 > 1$. Then $|y|_2 > 1$, so

$|y|_2 = |y|_1^c$ for some $c > 0$

For any $x \in K^*$, for any rational number $r/s \in \mathbb{Q}$,

$$|x|_1 < |y|_1^{r/s} \iff |x|_2 < |y|_2^{r/s}.$$

$$\Updownarrow$$

$$|x^r y^{-s}|_1 < 1 \iff |x^r y^{-s}|_2.$$

$$\implies \log_{|y|_1} |x|_1 = \log_{|y|_2} |x|_2$$

The approximation theorem

Let K be a field. Let $|\cdot|_1, \ldots, |\cdot|_n$ be pairwise inequivalent absolute values on K.

Then for any $a_1, \ldots, a_n \in K$ and $\varepsilon > 0$, $\exists\, x \in K$ s.t. $|x - a_1|_1 < \varepsilon \ldots\ldots |x - a_n|_n < \varepsilon$.

Pf for $n = 2$: $\exists\, \alpha \in K$ s.t. $|\alpha|_1 < 1$, $|\alpha|_2 \geq 1$

$\exists\, \beta \in K$ $|\beta|_1 < 1$, $|\beta|_2 \geq 1$

$y = \beta / \alpha$ $|y|_1 > 1$ $|y|_2 < 1$.

The sequence $\frac{y^n}{y^n + 1} = \frac{1}{1 + y^{-n}}$ converges to 1 under $|\cdot|_1$ and to 0 under $|\cdot|_2$ as $n \to \infty$.

Take $x = a_1 \cdot \frac{y^n}{y^n + 1} + a_2 \left(1 - \frac{y^n}{y^n + 1}\right)$ for $n >> 0$.

Approximation and the CRT

e.g. $K = \mathbb{Q}$ $\quad |\ |_2 \quad |\ |_3 \quad a_2, a_3 \in \mathbb{Z}$

$|x - a_2|_2 \leq 2^{-5}$

$|x - a_3|_3 \leq 3^{-7}$ $\quad x \in \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$

$\Rightarrow x \equiv a_2 \pmod{2^5}$

$x \equiv a_3 \pmod{3^7}$

$x \in \mathbb{Z}_{(2)} \Rightarrow$

$|x|_2 \leq 1$.

$|x - a_2|_2 \leq 2^{-5}$

$|a_2|_2 \leq 1$

Archimedean vs. nonarchimedean valuations

Eudoxus

$|\cdot|$ is archimedean if $\{|1|, |2|, |3|, \dots\}$ is unbounded
(on K) $\subseteq \mathbb{R}$

is nonarchimedean if the set is bounded.

(Triv. val gp, here)

prop $|\cdot|$ is nonarchimedean $\iff$ strong triangle inequality holds: for all $x, y \in K$
$|x+y| \le \max\{|x|, |y|\}$

pf if strong $\Delta \le$ holds, then $|n| \le 1$ $\forall n \in \mathbb{N}$. Conversely, suppose $|n| \le c$ $\forall n \in \mathbb{N}$.

then $|x+y| = |(x+y)^n|^{1/n} = \left|\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\right|^{1/n}$

$\le \left(\sum_{i=0}^{n} c|x|^i |y|^{n-i}\right)^{1/n}$ as $n \to \infty$, the upper bound tends to $\max\{|x|, |y|\}$

The strong triangle inequality on $\mathbb{Q}$

Let $|\cdot|$ be a nontrivial non-archimedean valuation on $\mathbb{Q}$.

Then $|n| \leq 1$ $\forall n \in \mathbb{N}$. If equality always holds, then get trivial valuation, so must exist p s.t. $|p| < 1$.

The set $\{n \in \mathbb{Z} : |n| < 1\}$ is an ideal containing $p\mathbb{Z}$.

so must be equal $\Rightarrow$ If $x \in \mathbb{Q}$, $|x|_p = 1$, then $|x| = 1$.

$\Rightarrow \exists c > 0$ s.t. $|x| = |x|_p^c$.

Ostrowski's classification of valuations of Q

Prop every valuation on $\mathbb{Q}$ is either

- trivial
- equivalent to $|\ |_p$ for some prime p

or – equivalent to $|\cdot|_\infty$ – real absolute value.

Pf only need to treat the archimedean case.

want to show: for $m, n \in \mathbb{Z}_{>1}$,

$$|m|^{1/\log m} = |n|^{1/\log n}.$$

($\Rightarrow$ equivalent to real absolute value.)

Proof of Ostrowski's theorem

write m in base n.

$$m = a_0 + \cdots + a_r n^r \qquad a_i \in \{0, \dots, n-1\}$$

$$|a_i| \le n$$

$r \le \log m / \log n$.

$$|m| \le \sum_{i=0}^{r} |a_i| |n|^i \le \underbrace{\left(1 + \frac{\log m}{\log n}\right) n}\ |n|^{\log m / \log n}$$

replace m with m^k for $k \gg 0$, do the same, take k-th root.

$$|m^k|^{1/k} \le \left[\underbrace{\left(1 + \frac{k \log m}{\log n}\right) n}\ |n|^{k \log m / \log n}\right]^{1/k}$$

$$\Rightarrow |m|^{1/\log m} \le |n|^{1/\log n}$$