# Extension of valuations

I am planning to post some supplemental notes related to this lecture.

# Complete fields

Let $L/K$ be a finite extension of fields

Fix a multiplicative valuation $|\cdot|_K$ on $K$

Q · Does this extend to a multiplicative valuation on $L$?

· If so, in how many ways?

e.g. $\mathbb{R} \to \mathbb{C}$

In this lecture assume $K$ is complete.

i.e. is there a absolute value $|\cdot|_L$ on $L$ s.t.

for all $x \in K$, $|x|_L = |x|_K$?

$|xy|_L = |x|_L |y|_L$

$|x|_L + |y|_L \geq |x+y|_L$

# Vector spaces over complete fields

Let $K$ be a field complete wrt absolute value $|\cdot|_K$

Let $V$ be a finite dimensional $K$-vector space.

Suppose given a function $|\cdot|_V : V \to \mathbb{R}$ s.t.

- $|v|_V \geq 0 \quad \forall v \in V$, $\quad |v|_V = 0 \iff v = 0$
- $|v_1 + v_2|_V \leq |v_1|_V + |v_2|_V$
- $|\lambda v|_V = |\lambda|_K |v|_V \quad \lambda \in K, v \in V$.

Then $V$ is complete wrt $|\cdot|_V$ in fact, if I pick a basis $e_1, \dots e_n$ of $V$, I get same topology as if I used

$$\lambda_1 e_1 + \dots \lambda_n e_n \longmapsto \max\{|\lambda_1|_K, \dots |\lambda_n|_K\}$$

# Vector spaces over complete fields

will show: $\exists K, k_2 > 0$ s.t. $k_2 |\ |_{sup} \le |\ |_v \le K |\ |_{sup}$

where $|\ |_{sup}$ is defined as on previous slide.

easy: $|\lambda_1 e_1 + \dots + \lambda_n e_n|_v \le |\lambda_1 e_1| \dots |e_n|_{sup} \max\{|e_1|_v \dots |e_n|_v\}$

$= \max\{|\lambda_i|\}$ $K$

induction on $n$:

$n = 1$: evident: $|\ |_v$ determined by its value on any one nonzero element.

$n > 1$: Pick $i \in \{1 \dots n\}$

Let $V_i$ = span of $e_1 \dots \hat{e_i} \dots e_n$ — by induction $V_i$ is complete wrt $|\ |_v$.

$\Rightarrow V_i$ is closed

$\Rightarrow V_i + e_i$ is also closed.

so $\exists\, \varepsilon_i > 0$ s.t. $\forall v \in V_i$ $0 \notin$

$|v + e_i| \ge \varepsilon_i$. Now if $\lambda_1 \dots \lambda_n \in K$

$|\lambda_i|$ is the biggest $\left|\dfrac{\lambda_1 e_1 + \dots + \lambda_n e_n}{\lambda_i}\right|_v \ge \varepsilon$.

$k_2 = \min_i \{\varepsilon_i\} > 0$

# Extension of a complete field: statement

non-trivial

Thm Let $K$ be a field complete w.r.t. abs value $|\cdot|_K$.

Let $L/K$ be a finite extension of fields.

Then there exists a unique absolute value $|\cdot|_L$ on $L$ extending $|\ |_K$

and moreover $L$ is complete with respect to $|\cdot|_L$.

[Keep in mind: if $K=\mathbb{Q}$, $L=\#$ field, start with $|\ |_p$ on $K$, I get extensions $|\ |_q$ for each prime $q$ of $L$ above $p$.]

# Extension of a complete field: completeness

Given an extension $||_L$

view $L$ as a finite dimensional $K$ vector space.

Then $|\lambda v|_L = |\lambda|_K |v|_L$ $\quad \lambda \in K$

$\underbrace{\phantom{|\lambda|_K}}_{= |\lambda|_L}$ $\quad v \in L$

$\Rightarrow L$ is complete.

# Extension of a complete field: uniqueness

Say $|\cdot|_{L,1}, |\cdot|_{L,2}$ are ~~two~~ distinct absolute values on $L$ extending $|\cdot|_K$.

They define the same topology.

So must be equivalent, i.e.

$\exists\, c>0$ s.t. $|x|_{L,2} = |x|_{L,1}^{c}$ $\forall x \in L$

But $\exists$ some $x \in K$ s.t. $|x|_K > 1$, $\Rightarrow c = 1$. $\Rightarrow\!\Leftarrow$.

# About archimedean completions: Ostrowski revisited

<u>Thm</u> (Ostrowski) Let $K$ be a field complete wrt an <u>archimedean absolute value</u>. Then $K = \mathbb{R}$ or $\mathbb{C}$.

<u>Pf Sketch</u>: $K$ <u>must</u> be of char $0$, so contains $\mathbb{Q}$.
By Ostrowski I, restriction of $|\ |_K$ to $\mathbb{Q}$ is equivalent to $|\cdot|_\infty$.

$\Rightarrow \mathbb{R} \subseteq K$

Next step: if $x \in K - \mathbb{R}$, then $x$ is quadratic over $\mathbb{R}$.

idea: look at $|x^2 - \mathrm{Tr}(z)x + \mathrm{Norm}(z)|_K$ as a function of $z \in \mathbb{C}$

Show this must take value $0$.

# Extension of a complete field: a candidate

Now suppose $K$ is given a nonarchimedean abs value.

If extension of $|\,|_K$ to $L$ exists, it must equal

$$|x|_L = |\mathrm{Norm}_{L/K}(x)|_K^{1/[L:K]}$$

(if $L/K$ Galois, uniqueness $\Rightarrow$ $\forall\, g \in \mathrm{Gal}(L/K)$

$|g(x)|_L = |x|_L$

b/c

$x \mapsto |g(x)|_L$ is also an abs value extending $|\,|_K$.)

$|xy|_L = |x|_L\,|y|_L$. $\quad |x|_L = 0 \Leftrightarrow x = 0$.

# Extension of a complete field: the triangle inequality?

Candidate $|x|_L = |\mathrm{Norm}_{L/K}(x)|_K^{1/[L:K]}$

Want this to satisfy

$$|x+y|_L \leq \max\{|x|_L, |y|_L\}.$$

WLOG assume $|x|_L \leq |y|_L \Rightarrow |x/y|_L \leq 1$

need to prove $\left|\frac{x}{y}+1\right|_L \leq 1$.

Let $P(T)$ be min poly of $x/y$ over $K$ (irreducible)

$= T^m + a_1 T^{m-1} + \dots + a_m$. Given $|a_m|_K \leq 1$

$\overset{??}{\Longrightarrow} |a_1|_K, \dots |a_{m-1}|_K \leq 1.$

# Preview of Hensel's lemma

- Hensel's lemma will tell us that many polynomials over K are actually reducible.