Extension of valuations

I am planning to post some supplemental notes related to this lecture.
Complete fields

Let $L/K$ be a finite extension of fields. Fix a multiplicative valuation $|\cdot|_L$ on $L$. 

Does this extend to a multiplicative valuation $|\cdot|_K$ on $K$? If so, in how many ways?

Q.5.: $\mathbb{R} \rightarrow \mathbb{C}$

In this lecture, assume $K$ is complete.

It is true on a boolean value $|\cdot|_K$ on $L$ such for all $x \in K$, $1 \cdot x^*_L = 1 \cdot x^*_K$?
Vector spaces over complete fields

Let $K$ be a field complete with absolute value $|\cdot|_K$.

Let $V$ be a finite-dimensional $K$-vector space.

Suppose given a function $|\cdot|_V: V \to \mathbb{R}$, such that:

1. $|v|_V \geq 0$ \quad $\forall v \in V$, $|v|_V = 0$ $\iff v = \mathbf{0}$

2. $|v, v_2|_V \leq |v|_V + |v_2|_V$

3. $|\alpha v|_V = |\alpha| \cdot |v|_V$ \quad $\forall \alpha \in K, v \in V$.

Then $V$ is complete with $|\cdot|_V$. In fact, if $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is a basis for $V$, set some triangle as if we used $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \longrightarrow \mathbf{e}_{n+1}, \mathbf{e}_{n+2}, \ldots, \mathbf{e}_k$.
Vector spaces over complete fields

will show: \( \exists k, k_2 > 0 \) s.t. \( k_2 \| v \| \leq \| v \| \) for all \( v \in V \).

where \( \| \cdot \| \) is defined as on previous slide.

Easy: \( 1 \| v \| = \| v \| \leq \| v \| \) for all \( v \in V \).

In fact:

\[ n \geq 1 : \text{consider } \| v \| \text{ determined by } \{ v_1, \ldots, v_n \} \]

\[ U + V_1 = \{ v + w : v \in U, w \in V_1 \} \] by induction on \( V_1 \).

So \( \exists \epsilon > 0 \) s.t. \( V_1 \supset V_i \) for all \( i \in I \) where \( V_i \) is complete.

\[ \| v + e_i \| \geq \epsilon_i \] for \( v \in V \) and \( e_i \) is the smallest \( \epsilon_i \).

\( k_2 = \min \{ \epsilon_i, i \geq 0 \} \)
Extension of a complete field: statement

Let $k$ be a field complete with a $\mathbb{R}$-valued norm $|\cdot|$. Let $L/k$ be a finite extension of fields. Then there exists a unique absolute value $|\cdot|_L$ on $L$ extending $|\cdot|_k$ and moreover $L$ is complete with respect to $|\cdot|_L$.

[Keep a norm $|\cdot|_p$ of $\mathbb{Q}$, $L = \mathbb{Q}$ field, start with $|\cdot|_p$ on $K$. I set extension $|\cdot|_L$ for each prime $\mathfrak{p}$ of $K$ above $p$.]
Extension of a complete field: completeness

Given a extension $\bar{L}/L$.

where $V$ has a finite dimensional $\bar{K}$-vector space.

then $\|V\|_L = \sup_{\|x\|_L = 1} \|x\|_V \leq \sup_{\|x\|_\bar{K} = 1} \|x\|_L$. 

$\forall \bar{K} \in \bar{L}$

$=)$ $L$ is complete.
Extension of a complete field: uniqueness

Say $|.|_{L}, |.|_{L_2}$ are the absolute values on $\mathbb{L}$ extending $|.|_K$. They are in the same topology, so must be equivalent, i.e.

$$\exists \epsilon > 0 \text{ s.t. } |x|_{L_2} = |x|_{L} \leq \epsilon \quad \forall x \in L$$

$$|x|_L = 1 \text{ for } |x|_K > 1 \implies c = 1 \implies \exists \epsilon.$$
About archimedean completions: Ostrowski revisited

Then (Ostrowski) let $K$ be a field complete with an archimedean absolute value. Then $K = \mathbb{R}$ or $\mathbb{C}$.

Sketch: $K$ must be of char 0, so contains $\mathbb{Q}$. By Ostrowski, restriction of $|\cdot|_K$ to $\mathbb{Q}$ is equivalent to $|\cdot|_\infty$.

$\Rightarrow$ $\mathbb{R} \subseteq K$

Next: $x \not\in \mathbb{R}$, then $x$ is quadratic over $\mathbb{R}$.

Idea: look at $x^2 - \tau(x)(2)x + \nu(x)(2)$ in $K$ show this must take value 0.
Extension of a complete field: a candidate

Now suppose $K$ is a complete field and $\mathbb{A}_F$ is its absolute value. If $\mathbb{A}_F$ contains $K$, then, it must be that $x_1 = |\text{Num}_{\mathbb{A}_F/K}(x)|_{\mathbb{A}_F}$.

For $L/K$, consider $\text{Ext}_{L/K}(x)$.

- If $L/K$ contains $\mathbb{A}_F$, then $x_1 \in \mathbb{A}_F$.
- If $x_1$, then $g(x)_{L} = x_{\mathbb{A}_F}$ is a complete extension of $|x| L$.
- $1_{x_{L}} = 1 \times 1_{L} / y_{L}$. $1_{x_{L}} = 0 \Leftrightarrow x = 0$. 

(Hand-drawn annotations:)

\[ 1_{x_{L}} = |\text{Num}_{\mathbb{A}_F/K}(x)|_{K} \]
Extension of a complete field: the triangle inequality?

\[ \text{Let } x, y \in \mathbb{K}, \quad |x|_\mathbb{K} = \text{Max} \{ |x|_m, |x|_n, \ldots, |x|_m \} \]

\[ |x + y|_\mathbb{K} \leq \text{Max} \{ |x|_\mathbb{K}, |y|_\mathbb{K} \} \]

We can assume \( |x|_\mathbb{K} \leq |y|_\mathbb{K} = 1 \) \( |x/y|_\mathbb{K} \leq 1 \).

\[ |1/\mathbb{K} + 1|_\mathbb{K} \leq 1 \cdot \]

At \( \mathbb{F}(T) \) be minimal of \( \mathbb{K} \) are \( \mathbb{K} \) (irreducible)

\[ T^m + a_1 T^{m-1} + \ldots + a_m. \quad \text{Given } \lambda \text{ are } \lambda \mathbb{K} \leq 1 \]

where \( \lambda_0, \lambda_1, \ldots, \lambda_{m-1} \mathbb{K} \leq 1 \).
Hensel's lemma will tell us that many polynomials over $K$ are actually reducible.