# Newton polygons

Last lecture this Friday (December 11). Regular office hours also end this week, but I plan to hold some sporadic office hours after that (see Zulip).

Course evaluations due Monday, December 14.

I can accept late homeworks through Friday, December 18. (Grades are due December 22).

Math 204B begins Monday, January 4 (new home page, same Zoom, same Zulip).

# Reminder: Hensel's lemma and irreducibility

$K$ = field complete with a nonarchimedean absolute value

$\mathcal{O}_K = \{\alpha \in K : |\alpha| \leq 1\}$ subring

$\mathfrak{p} = \{\alpha \in K : |\alpha| < 1\}$ unique maximal ideal

$P(x) \in \mathcal{O}_K[x]$ irreducible (over $K$)

$P(x) = a_n x^n + \cdots + a_0$. Then $\max\{|a_i|\}$

$= \max\{|a_0|, |a_n|\}$

idea: renormalize so that $\max\{|a_i|\} = 1$

apply Hensel's lemma.

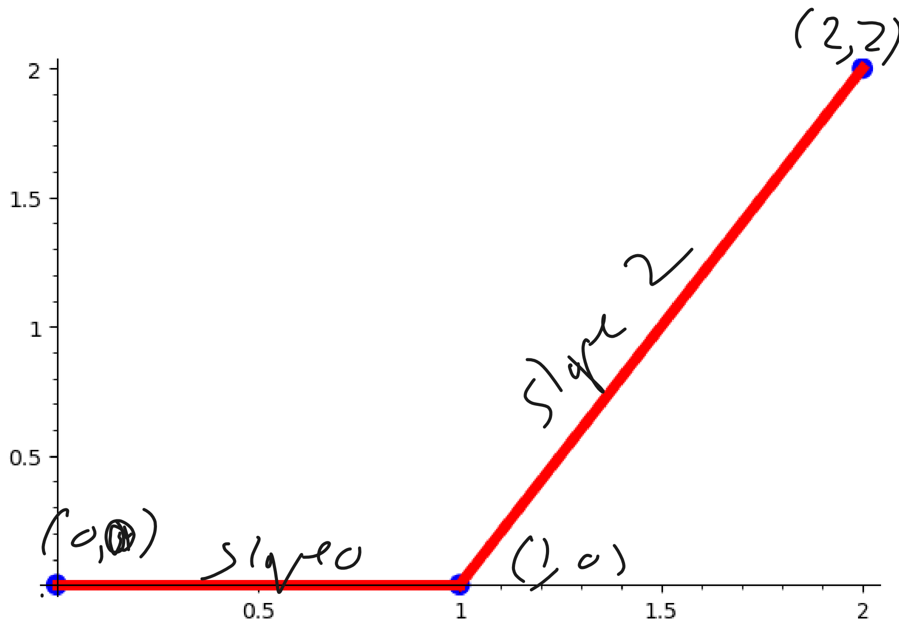
# Three examples of Newton polygons: example 1

① $p(x) = (x-1)(x-p^2)$

$v_p = 0 \qquad v_p = 2$

$\underbrace{1}_{v_p=0} \cdot x^2 - \underbrace{(1+p^2)}_{v_p=0} x + \underbrace{p^2}_{v_p=2}$

Newton: $\mathbb{C}((z))$

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sage: points([(0,0), (1,0), (2,2)], size=100) + plot(0, (0,1), color="red", thickness=5) + plot(2*(x-1), (1,2), color="red", thickness=5)
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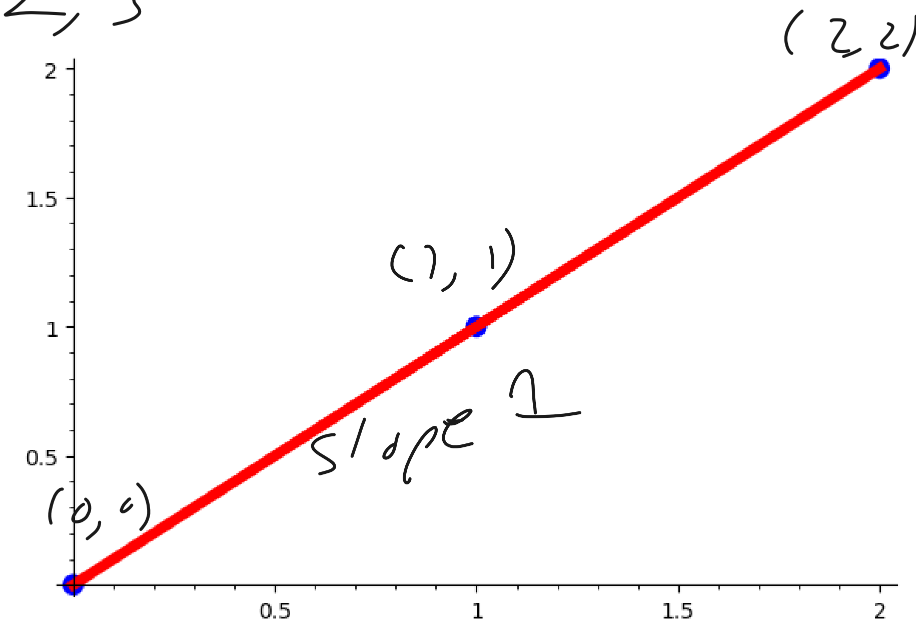
# Three examples of Newton polygons: example 2

$p \neq 2, 3$

$(x-p)(x-2p)$

$v_p = 1 \qquad v_p = 1$

$= 1x^2 - 3px + 2p^2$

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sage: points([(0,0), (1,1), (2,2)], size=100) + plot(x, (0,2), color="red", thickness=5)
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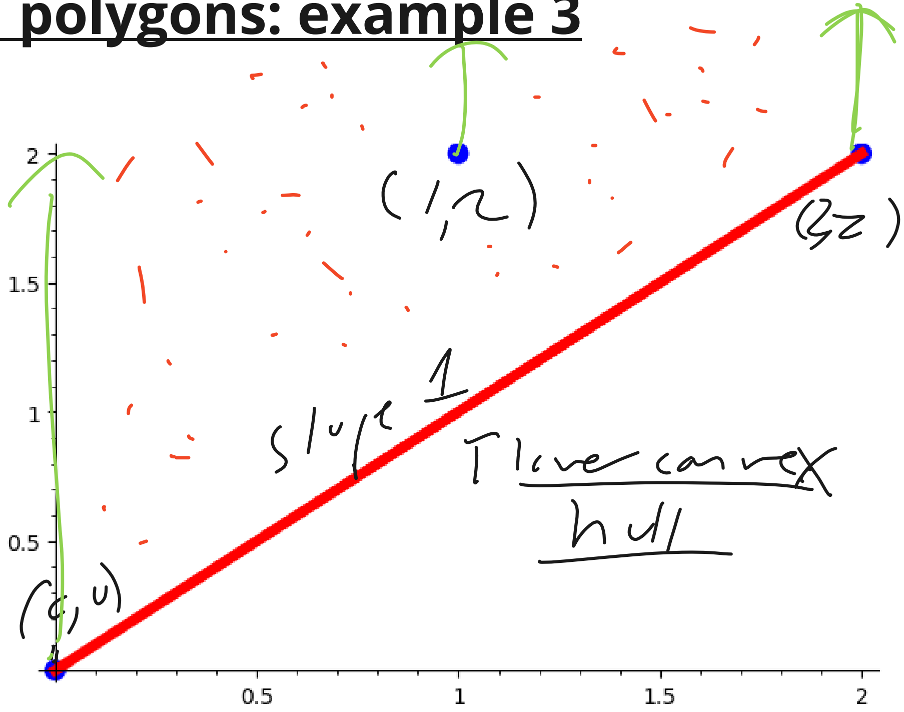
# Three examples of Newton polygons: example 3

$(x-p)(x-(p-1)p)$

$v_p = 1 \qquad v_p = 1$

$= 1x^2 - p^2x + p^2(p-1)$

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sage: points([(0,0), (1,2), (2,2)], size=100) + plot(x, (0,2), color="red", thickness=5)
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# Newton polygons: the general definition

$K$ = field with an additive valuation $v$

$v(x+y) \geq \min\{v(x), v(y)\}$

$v(xy) = v(x) + v(y)$

assume $a_n \neq 0$, $a_0 \neq 0$

$P(x) = a_n x^n + \cdots + a_0 \in K$

Draw the picture:

take points: $\{(i, v(a_{n-i})) : i = 0, \ldots n\}$

$a_{n-i} \neq 0$

take the lower convex hull:

this is a polygonal line from $(0, v(a_n))$ to $(n, v(a_0))$.

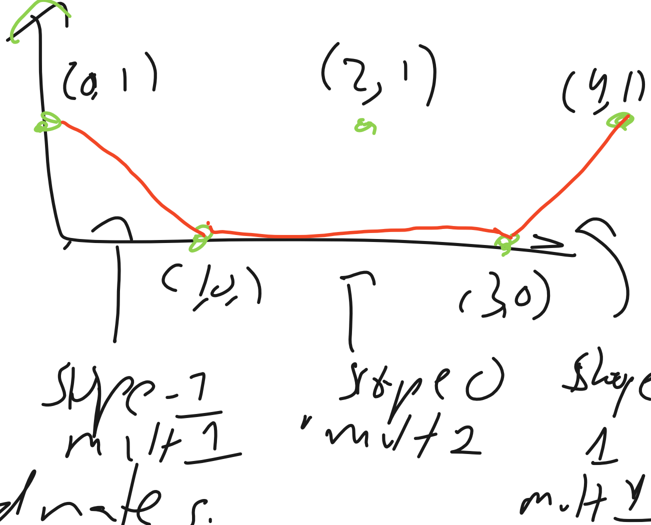
# Slopes and multiplicities

The Newton polygon is the graph of a piecewise affine function
$[0, n] \to \mathbb{R}$.

Each slope occurs between some pair of integer x-coordinates.

These form a multiset where each slope has multiplicity = width of the corresponding line segment.

slope multiset = $\{-1, 0, 0, 1\}$



# The main theorem

For $K$ as before

suppose monic $p(x) \in K[x]$ factors completely into linear polynomials (and $x \nmid p(x)$)

$$(x-\alpha_1)\cdots(x-\alpha_n)$$

then

multiset $\{v(\alpha_1), \ldots, v(\alpha_n)\}$

equals the slope multiset of Newton polygon.

(if $p$ does not factor over $K$, so to an extension field choose an extension of valuation, work there)

# Proof of the theorem: comparison of polygons

Consider Newton polygon for with $\{v(\alpha_1), \ldots, v(\alpha_n)\}$ and slope multiset.

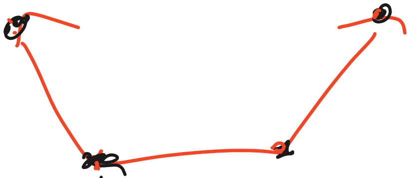

Step 1: if I draw a polygon based on roots, then every point $(i, v(a_{n-i}))$ lies on or above this.

$v(\alpha_1) \le v(\alpha_2) \le \cdots \le v(\alpha_n)$

For this putative Newton polygon, the point with x-coord $i$ has y-coord $v(\alpha_1) + \cdots + v(\alpha_i) = v(\alpha_1 \cdots \alpha_i)$

whereas $a_{n-i} = \pm \sum_{j_1 < \cdots < j_i} \alpha_{j_1} \cdots \alpha_{j_i}$ smallest valuation of an $i$-fold product of roots.

# Proof of the theorem: matching of vertices



← determined by $v(\alpha_1) \dots v(\alpha_n)$

$(i, v(\alpha_1) + \dots + v(\alpha_i))$ where $v(\alpha_i) < v(\alpha_{i+1})$

Claim: we get $v(a_{n-i}) = v(\alpha_1) + \dots v(\alpha_i)$

Because in expression $a_{n-i} = \sum \alpha_{j_1} \cdots \alpha_{j_i}$, $j_1 < \dots < j_i$

unique dominant term $\alpha_1 \cdots \alpha_i$

# The Eisenstein irreducibility criterion revisited

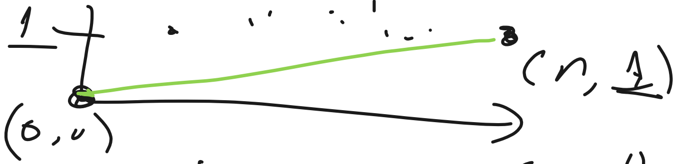
$P(x) \in \mathbb{Z}[x]$ $\qquad p = \text{prime}$

$= a_n x^n + \dots + a_0$

$p \nmid a_n, \quad p \mid a_{n-1}, \dots, a_0, \quad p^2 \nmid a_0$

$\Longrightarrow$ $P(x)$ irreducible over $\mathbb{Q}$.

Newton polygon




has slope $\frac{1}{n}$ with mult $n$

In fact $P(x)$ irreducible over $\mathbb{Q}_p$!!

otherwise $P(x) = Q(x)R(x)$ where $\deg Q, \deg R < n$

slopes of NP's of $Q$ & $R$ have denominators $< n$.