The Kronecker-Weber theorem: a preview of Math 204B

Last lecture this Friday (December 11). Regular office hours also end this week, but I plan to hold some sporadic office hours after that (see Zulip).

Course evaluations due Monday, December 14.

I can accept late homeworks through Friday, December 18. (Grades are due December 22.)

Math 204B begins Monday, January 4 (new home page, same Zoom, same Zulip).

This lecture and the next are based on my notes on class field theory: https://kskedlaya.org/cft.

The Kronecker-Weber theorem: statement

Thm Let K be a number field which is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q})$ abelian (i.e. K is an abelian extension of $\mathbb{Q}$)

Then $K \subseteq \mathbb{Q}(\zeta_n)$ for some n.

e.g. every quadratic extension has this property (Gauss)

Artin maps for abelian number fields

Let $K/\mathbb{Q}$ be an abelian extension. By K-W, $K \subseteq \mathbb{Q}(\zeta_n)$ for some n.

abelian

$$(\mathbb{Z}/n\mathbb{Z})^* \cong \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(K/\mathbb{Q})$$

$$a \longmapsto (\zeta_n \mapsto \zeta_n^a)$$

depends only on p

Let p be a rational prime not dividing n. Then p does not ramify in $\mathbb{Q}(\zeta_n)$ or K, pick a prime $\mathfrak{q}$ of K above p, then $G_{\mathfrak{q}}$ is cyclic generated by $\mathrm{Frob}_{\mathfrak{q}}$.

$$\boxed{\mathrm{Frob}_{\mathfrak{q}} \in \mathrm{Gal}(K/\mathbb{Q}) \text{ is image of } p \in (\mathbb{Z}/n\mathbb{Z})^*}$$

Artin reciprocity law for $\mathbb{Q}$.

A further preview: the Artin reciprocity law

For this slide only, let L/K any abelian extension of number fields

Artin reciprocity law: there is a similar relationship between $\mathrm{Gal}(L/K)$ and some group of ideal classes (i.e. some quotient of J_K defined by a congruence condition on principal ideal) which predicts Frobenius elements for unramified primes of K

A further preview: the existence theorem

The existence theorem will assert that the composition of all abelian extensions of K is "as big as possible given Artin reciprocity".

However in general it will not be an explicit extension.

(K = imaginary quadratic - get explicit abelian extensions from CM elliptic curves

K = function fields - get abelian extensions from Drinfeld modules)

K = local field - Lubin-Tate theory

The local Kronecker-Weber theorem: statement

Thm Let p be a prime.
Let $K/\mathbb{Q}_p$ be a finite abelian extension.
Then $K \subseteq \mathbb{Q}_p(\zeta_n)$ for some n.

(There is an analogue of Artin reciprocity, but it's not yet clear how to formulate it.)
Better formulation will be via adeles

Reduction to the local case

Thm local K-W (for all p) $\Rightarrow$ K-W.

Pf Let K be an abelian extension of $\mathbb{Q}$.

For each prime p of $\mathbb{Q}$ that ramifies in K,

apply local K-W to write $K_{\mathfrak{p}} \subseteq \mathbb{Q}_p(\zeta_{n_p})$

(for $\mathfrak{p}$ some prime above p)

Let $n = \prod_p p^{v_p(n_p)}$. We claim $K \subseteq \mathbb{Q}(\zeta_n)$

(i.e. $K(\zeta_n) = \mathbb{Q}(\zeta_n)$)

Reduction to the local case

$L = K(\zeta_n)$

I_p = inertia group of p in L (i.e. inertia group of a prime of L above p)

let U = maximal unramified extension of $L_{\mathfrak{q}}$ over $\mathbb{Q}_p$. Then $U(\zeta_{p^*}) = L_{\mathfrak{q}}$ ($* = v_p(n)$)

(using local K-W on $L_{\mathfrak{q}}$)

$\Rightarrow I_p = \mathrm{Gal}(L_{\mathfrak{q}}/U) \subseteq \left(\mathbb{Z}/p^{v_p(n)}\mathbb{Z}\right)^*$

Let I be the group generated by all of the I_p;

then $|I| \leq \prod_p |I_p| \leq \prod_p \phi(p^{v_p(n)}) = \phi(n) = [\mathbb{Q}(\zeta_n):\mathbb{Q}]$

meanwhile, fixed field of I is everywhere unramified$/\mathbb{Q}$

$\overset{?}{\Rightarrow}$ (Minkowski) is equal to $\mathbb{Q}$. so $I = \mathrm{Gal}(L/\mathbb{Q})$

$\Rightarrow [L:\mathbb{Q}] \leq [\mathbb{Q}(\zeta_n):\mathbb{Q}] \Rightarrow L = \mathbb{Q}(\zeta_n)$.

Kummer extensions

Let n be a positive integer

Let K be a field of characteristic not dividing n.

For every $a \in K^*$, $K(a^{1/n})$ is some field extension of K;

Now let's assume $\zeta_n \in K$
primitive n-th root of unity.

Then $K(a^{1/n})$ is a Galois extension of K
with Gal group some subgroup of $\mathbb{Z}/n\mathbb{Z}$

$$c \in (\mathbb{Z}/n\mathbb{Z})^* \to \left[a^{1/n} \to a^{1/n} \zeta_n^c \right]$$

$\mathrm{Gal}(K(a^{1/n})/K) = (\mathbb{Z}/n\mathbb{Z})^* \Longleftarrow a \notin (K^*)^p$ for every prime factor p of n.

The Kummer pairing

assume $\zeta_n \in K$

$$\mathrm{Gal}(\overline{K}/K) \times K^* \longrightarrow \langle \zeta_n \rangle \subseteq K^*$$

$\langle \zeta_n \rangle \cong \mathbb{Z}/n\mathbb{Z}$

$$(\sigma, a) \longmapsto c$$

where $\dfrac{\sigma(a^{1/n})}{a^{1/n}} = c$

this does not depend on choice of $a^{1/n} \in \overline{K}$

Kummer's theorem (statement) and "Kummer theory"

Assume $\zeta_n \in K$

<u>Theorem</u> The Kummer pairing induces an isomorphism

$$K^\times/(K^\times)^n \xrightarrow{\sim} \mathrm{Hom}_*(\mathrm{Gal}(\bar{K}/K), \underbrace{\langle \zeta_n \rangle}_{\mu_n})$$

$*$ which factor through $\mathrm{Gal}(L/K)$

(i.e. <u>continuous for</u> profinite topology on $\mathrm{Gal}(\bar{K}/K)$

e.g. $n=2$