The local Kronecker-Weber theorem

Last lecture of Math 204A. Regular office hours also end this week, but I plan to hold some sporadic office hours after that (see Zulip).

Course evaluations due Monday, December 14.

I can accept late homeworks through Friday, December 18. (Grades are due December 22.)

Math 204B begins Monday, January 4 (new home page, same Zoom, same Zulip).

This lecture and the previous one are based on my notes on class field theory: https://kskedlaya.org/cft.
(Keep in mind that I will be revising these as I go over them in Math 204B.)

The local Kronecker-Weber theorem

Fix prime p

Thm Let $K/\mathbb{Q}_p$ be a finite abelian extension. Then $K \subseteq \mathbb{Q}_p(\zeta_n)$ for some positive integer n.

recall this implies corresponding statement for $\mathbb{Q}$ (Kronecker-Weber theorem) by considering ramification and using Minkowski's theorem that there are no nontrivial extensions of $\mathbb{Q}$ unramified everywhere.

Tame vs. wild

$\mathrm{Gal}(K/\mathbb{Q}_p) \cong \prod_i \mathbb{Z}/q_i^{r_i}\mathbb{Z}$

correspondingly, K is a composition of abelian extensions of prime-power degree. So I can assume

$\mathrm{Gal}(K/\mathbb{Q}_p) = \mathbb{Z}/q^r\mathbb{Z}$, q prime

Tame: case $q \neq p$

$\Rightarrow e(K/\mathbb{Q}_p) =$ coprime to p

Wild: case $q = p$

$\Rightarrow e(K/\mathbb{Q}_p) =$ power of p

Unramified (local) extensions are cyclotomic (and Galois!)

let L/K be finite extensions of $\mathbb{Q}_p$

assume L/K is unramified, i.e. $e(L/K) = 1$

($\text{i.e.}, e(\mathfrak{q}_L/\mathfrak{p}_K) = 1$)

<u>lemma</u> In this case, $L = K(\zeta_{q-1})$ where $q = \#$ residue field of L.

in particular, L/K is Galois and abelian

<u>Pf</u> $\Phi_{q-1}(x)$ has a root in the residue field of L and is separable, so Hensel's lemma implies $K(\zeta_{q-1}) \subseteq L$. (and vice versa)

idea: an unramified extension is <u>uniquely determined</u> by its residue field

Totally tamely (local) ramified extensions are Kummer

Lemma Let L/K be a finite totally tamely ramified extension of finite extensions of $\mathbb{Q}_p$

$(f(L/K)=1,\ \underset{=e}{e}(L/K)=\text{coprime to } p)$

Then $L=K(\pi^{1/e})$ for some uniformizer π of K

($\Rightarrow$ if $\zeta_e\in K$, then get a Galois extension)

Pf Let π_L be a uniformizer of L, write $\pi_L^e\mathcal{O}_L=\pi\cdot\mathcal{O}_L$

Choose π so that $\pi_L^e=\pi\cdot u$ where $u\in\mathcal{O}_L^\times$

$\equiv 1 \bmod \pi_L$

then $u^{1/e}\in\mathcal{O}_L$ by Hensel's lemma (or binomial series) $\Rightarrow K(\pi^{1/e})\subseteq L$

$=$ by comparing degrees.

See also Neukirch Proposition II.7.7.

Local Kronecker-Weber: the tame case

q prime
$q \neq p$
$r \geq 1$

Let $K/\mathbb{Q}_p$ be abelian $\mathrm{Gal}(K/\mathbb{Q}_p) \cong \mathbb{Z}/q^r\mathbb{Z}$

$$\begin{array}{l} K \\ \mid \ \text{totally tamely ramified of degree } e = e(K/\mathbb{Q}_p) \\ L \\ \mid \ \text{unramified} \\ \mathbb{Q}_p \end{array}$$

we know $L \subseteq \mathbb{Q}_p(\zeta_n)$ for some n.

write $K = L(\pi^{1/e})$ for some uniformizer π of L.

note: $\frac{\pi}{p} \in \mathcal{O}_L^*$, so $\pi = cu$ $u \in \mathcal{O}_L^*$ $c \in p\mathbb{Z}_p^*$ s.t.

$\boxed{\text{note: } c = -p}$

$\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p(c^{1/p-1})$

$K = L((cu)^{1/e})$ so $K(u^{1/e}) = K(c^{1/e})$

$\subseteq K \cdot \mathbb{Q}_p(c^{1/e})$.

Local Kronecker-Weber: the tame case

$$\underbrace{L(u^{1/e})/L}_{\text{unramified!}}/\underbrace{L/\mathbb{Q}_p}_{\text{unramified}} \longrightarrow L(u^{1/e})/\mathbb{Q}_p \text{ unramified}$$

$\Rightarrow$ cyclotomic!

$L(u^{1/e}) \subseteq \mathbb{Q}_p(\zeta_m)$

$$\mathbb{Q}_p(c^{1/e}) \subseteq \underbrace{K}_{\text{abelian}} \cdot \underbrace{L(u^{1/e})}_{\text{abelian}}$$

so $\mathbb{Q}_p(c^{1/e})$ is $\underline{\text{abelian}}$.

$\Rightarrow$ below

$$\Rightarrow \zeta_e \in \mathbb{Q}_p(c^{1/e}) \Rightarrow \zeta_e \in \mathbb{F}_p \Rightarrow e \mid p-1.$$

residue field $\mathbb{F}_p$

$$\Rightarrow K \subseteq \mathbb{Q}_p(\zeta_{\underline{n}mp}).$$

Plan for the wild case: p odd

$p \neq 2$

$\mathrm{Gal}(K/\mathbb{Q}_p) \cong \mathbb{Z}/p^r\mathbb{Z}$

two other extensions with this property are

$K_1 = \mathbb{Q}_p(\zeta_{p^{p^r}-1})$ unramified and $K_2 =$ index $p-1$ subfield of $\mathbb{Q}_p(\zeta_{p^{r+1}})$

Claim: $K \subset K_1 K_2$

Suppose otherwise; then $\mathrm{Gal}(KK_1K_2/\mathbb{Q}_p)$

$\cong (\mathbb{Z}/p^r\mathbb{Z})^2 \times (\mathbb{Z}/p^s\mathbb{Z})$, $1 \leq s \leq r$

$\Rightarrow K$ contains a subextension with Galois group $(\mathbb{Z}/p\mathbb{Z})^3$.

$\Longrightarrow$ reduce to $r=1$ case!

Plan for the wild case: p odd

plus

solve at $K/\mathbb{Q}_p$ ($p \neq 2$)

$\mathrm{Gal}(K/\mathbb{Q}_p) \cong (\mathbb{Z}/p\mathbb{Z})^3$

Strategy:

$K(\zeta_p)$ has Galois group $(\mathbb{Z}/p\mathbb{Z})^3 \times (\mathbb{Z}/(p-1)\mathbb{Z})$

and $K(\zeta_p)/\mathbb{Q}_p(\zeta_p)$ is a composition of Kummer extensions

i.e. $K(\zeta_p) = \mathbb{Q}_p(\zeta_p)(a^{1/p}, b^{1/p}, c^{1/p})$

for some $a, b, c \in \mathbb{Q}_p(\zeta_p)^*$

— first analyze $\mathbb{Q}_p(\zeta_p)^* / (\mathbb{Q}_p(\zeta_p)^*)^p \ni \bar{a}, \bar{b}, \bar{c}$

— use fact that we started with a Galois abelian extension over $\mathbb{Q}_p$

(in particular, if $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$, $K(\zeta_p) = \mathbb{Q}_p(\zeta_p)(\sigma(a)^{1/p}, \sigma(b)^{1/p}, \sigma(c)^{1/p})$)

Plan for the wild case: p even

$p = 2$

$K/\mathbb{Q}_p$ abelian, $\mathrm{Gal}(K/\mathbb{Q}_p) \cong \mathbb{Z}/2 \times \mathbb{Z}$

$K_1 = \mathbb{Q}_p(\zeta_{p^r - 1})$ unramified $\quad K_2 = \mathbb{Q}_p(\zeta_{2^{r+2}})$

$\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^r\mathbb{Z}$

So there is an extension with

$\mathrm{Gal}(K/\mathbb{Q}_2) \cong (\mathbb{Z}/2\mathbb{Z})^3 \quad (K = \mathbb{Q}_2(\zeta_{24}))$

but not with $\cong (\mathbb{Z}/2\mathbb{Z})^4$

Plan for the wild case: p even

similar to previous case, $(\mathbb{Z}/2\mathbb{Z})^4$

rule out $(\mathbb{Z}/4\mathbb{Z})^3$