Math 204A (Number Theory), UCSD, fall 2020
Problem Set 2 – due Thursday, October 22, 2020

For the Chinese remainder theorem for Dedekind domains, see for example Neukirch Theorem I.3.6.

1. Let R be a Dedekind domain. Prove that for every nonzero ideal I of R, R/I is a principal ideal ring. (Hint: use the Chinese remainder theorem to reduce to the case where I is a power of a prime $\mathfrak{p}$. In that case, choose $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ and consider the powers of π.)

2. Let R be a Dedekind domain. Prove that every ideal of R can be generated by at most two elements.

3. Let R be the ring $\mathbb{Z}[\alpha]/(\alpha^3 - \alpha - 1)$. Show that the prime factorization of the principal ideal $23R$ is given by
$$23R = (23, \alpha - 10)^2(23, \alpha - 3).$$
(In particular, you should show that the factors are indeed prime.)

4. Prove that the constant 2^n in Minkowski's lattice point theorem cannot be improved.

5. Let K be a number field. Using the finiteness of the class group of K, prove that there exists a finite extension L of K such that every ideal of $\mathcal{O}_K$ generates a principal ideal of $\mathcal{O}_L$.

6. Let $p > 2$ be a prime number and put $K = \mathbb{Q}(\zeta_p)$.
   (a) Compute $\mathrm{Trace}_{K/\mathbb{Q}}(\zeta_p^j)$ for $j = 0, \ldots, p-1$.
   (b) Compute $\mathrm{Norm}_{K/\mathbb{Q}}(1 - \zeta_p)$.
   (c) Show that $(1 - \zeta_p)\mathcal{O}_K \cap \mathbb{Z} = p\mathbb{Z}$. (Note: we will use this later to show that $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$, so don't assume this here.)