Math 204A (Number Theory), UC San Diego, fall 2022
Problem Set 2 – due Thursday, October 13, 2022

Submit *at most five* of the listed problems.

1. Use the L-Functions and Modular Forms Database to identify the first degree-3 number field K (sorted by absolute discriminant) whose ring of integers is reported to be not monogenic. Then prove this by showing that for any $\theta \in \mathfrak{o}_K$, the index $[\mathfrak{o}_K : \mathbb{Z}[\theta]]$ is divisible by 2.

2. Let R be a Dedekind domain. Prove that for every nonzero ideal I of R, R/I is a principal ideal ring. (Hint: use the Chinese remainder theorem to reduce to the case where I is a power of a prime $\mathfrak{p}$. In that case, choose $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ and consider the powers of π.)

3. Let R be a Dedekind domain. Prove that every ideal of R can be generated by at most two elements.

4. Let R be an integral domain in which every nonzero ideal admits a unique factorization into prime ideals. Prove that R is a Dedekind domain.

5. Let R be the ring $\mathbb{Z}[\alpha]/(\alpha^3 - \alpha - 1)$. Show that the prime factorization of the principal ideal $23R$ is given by
$$23R = (23, \alpha - 10)^2(23, \alpha - 3).$$
(In particular, you should show that the factors are indeed prime.)

6. Read the description given in `https://math.stackexchange.com/questions/2151591/is-the-relative-discriminant-a-principal-ideal` of an extension L/K of number fields whose relative discriminant is not principal, then find another example where K is a *real* quadratic field.

7. Let $p > 2$ be a prime number and put $K = \mathbb{Q}(\zeta_p)$.
   (a) Compute $\mathrm{Trace}_{K/\mathbb{Q}}(\zeta_p^j)$ for $j = 0, \dots, p-1$.
   (b) Compute $\mathrm{Norm}_{K/\mathbb{Q}}(1 - \zeta_p)$.
   (c) Show that $(1 - \zeta_p)\mathfrak{o}_K \cap \mathbb{Z} = p\mathbb{Z}$. (Note: we will use this later to show that $\mathfrak{o}_K = \mathbb{Z}[\zeta_p]$, so don't assume this here.)

8. Let K be a number field. Using the finiteness of the class group of K, prove that there exists a finite extension L of K such that every ideal of $\mathcal{O}_K$ generates a principal ideal of $\mathcal{O}_L$. (This is related to the etymology of the term "ideal": an ideal is meant to be the set of multiples of an "ideal number" which happens not to be present in K.)