Math 204A (Number Theory), UC San Diego, fall 2022
Problem Set 3 – due Thursday, October 20, 2022

Submit *at most five* of the listed problems.

1. Let R be a Dedekind domain. Prove that the following statements are equivalent.

   (a) The ring R is a principal ideal domain.

   (b) The ring R is a unique factorization domain.

   (c) The class group of R is trivial.

   (Hint: for (b) $\implies$ (c), it is enough to show that every prime ideal $\mathfrak{p}$ is principal. First show that $\mathfrak{p}$ contains an irreducible element α, then show that α generates a nonzero prime ideal.)

2. Let L/K be a nontrivial extension of number fields such that $[\mathfrak{o}_L^\times : \mathfrak{o}_K^\times]$ is finite. Prove that K is a totally real field (i.e., all of its archimedean embeddings are real) and $L = \mathbb{Q}(\sqrt{\alpha})$ for some $\alpha \in K$ such that $\tau(\alpha) < 0$ for every embedding $\tau : K \to \mathbb{R}$. Such a field L is called a *CM field* (for "complex multiplication").

3. Let K be a number field of degree n with signature (r_1, r_2). We derive Minkowski's improved estimate for the minimum norm of an ideal in an ideal class of K.

   (a) Prove that for any $t > 0$, the region

   $$X = \{(z_\tau) \in K_\mathbb{R} : \sum_\tau |z_\tau| < t\}$$

   has volume $2^{r_1}\pi^{r_2}\frac{t^n}{n!}$. (Hint: it may be easiest to set this up as a multiple integral, using polar coordinates for each complex embedding.)

   (b) Show that any nonzero ideal I of $\mathfrak{o}_K$ contains a nonzero element α satisfying

   $$\left|\mathrm{Norm}_{K/\mathbb{Q}}(\alpha)\right| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2}\sqrt{|d_K|}[\mathfrak{o}_K : I].$$

   (Hint: apply Minkowski's lattice point theorem to the region X from (a), then use the arithmetic-geometric mean inequality.)

   (c) Deduce (as in lecture) that every ideal class of K is represented by some integral ideal of norm at most $\frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2}\sqrt{|d_K|}$.

4. Let K be a number field for which $|d_K| = 1$. Prove that $K = \mathbb{Q}$. (Hint: show that $\frac{n!}{n^n}\left(\frac{4}{\pi}\right)^n < 1$ for $n > 1$, then apply the previous exercise.)

5. Use the improved Minkowski estimate to show that $\mathbb{Q}(\sqrt{11})$ has class number 1.

6. Let K be the number field $\mathbb{Q}[x]/(x^3 - x^2 + x + 1)$.

   (a) Find the home page of K in the LMFDB.

   (b) Report what the LMFDB says about the signature of K and the structure of the group $\mathfrak{o}_K^\times$ (including generators).

   (c) Prove that this answer is correct.

7. Repeat part (a) and (b) of the previous exercise for the fields

$$\mathbb{Q}(2^{1/3}), \qquad \mathbb{Q}(\sqrt{3}, \sqrt{5}), \qquad \mathbb{Q}(\zeta_5).$$

   (Hint: it may help to filter by signature and/or Galois group.)

8. (a) Use SageMath to list all the imaginary quadratic fields with discriminant in $[0, 10^4]$ with class number 1. The fact that there are no others of *any* discriminant is a deep theorem (the resolution of the Gauss class number one problem by Baker–Heegner–Stark).

   (b) Compute the fraction of real quadratic fields with discriminant in $[-10^4, 0]$ with class number 1. Note that this is quite far from zero!

   (c) Make a plot comparing two sets of data: the class number of an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ as a function of D, and the class number times the logarithm of a fundamental unit of a real quadratic field $\mathbb{Q}(\sqrt{D})$ as a function of D. These should look much more similar.