Math 204A (Number Theory), UC San Diego, fall 2022
Problem Set 7 – due Thursday, November 17, 2022

Submit *at most five* of the listed problems.

1. (a) Prove that for any finite extension K of $\mathbb{Q}_p$, the group of roots of unity in K is finite. (Hint: separate into the p-part and the prime-to-p part.)
   (b) Compute this group for $K = \mathbb{Q}_p$.
2. (a) Compute the group $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$.
   (b) Use this to show that every quadratic extension of $\mathbb{Q}_2$ is contained in $\mathbb{Q}_2(\zeta_{24})$. This is an important partial result towards the *local Kronecker–Weber theorem*, an initial step in class field theory.
3. Let K be a field complete with respect to a nonarchimedean absolute value. Let $P(x) \in \mathfrak{o}_K[x]$ be a polynomial and suppose $\alpha \in \mathfrak{o}_K$ satisfies
$$|P(\alpha)| < |P'(\alpha)|^2.$$
Show that the *Newton–Raphson iteration*
$$\alpha_0 = \alpha, \qquad \alpha_{n+1} = \alpha_n - \frac{P(\alpha_n)}{P'(\alpha_n)}$$
converges to a root of P in K.
4. Neukirch, exercise II.5.5 (corrected): Let K be a finite extension of $\mathbb{Q}_p$. Prove that as n varies over positive integers, the subgroup $(\mathfrak{o}_K^\times)^n$ of $K^\times$ form a basis of neigborhoods of 1 in K.
5. Let K be a field equipped with (but not necessarily complete with respect to) a nonarchimedean absolute value. Let L be any field containing K (not necessarily a finite extension). Prove that there exists at least one extension of the absolute value on K to an absolute value on L. (Hint: use Zorn's lemma to reduce to the case where $L = K(\alpha)$ where α is either algebraic or transcendental over K.)
6. In this exercise, we give Monsky's amazing proof of the following theorem: a square in the Euclidean plane cannot be dissected into an *odd* number of triangles, all of the same area.

   (a) Use the previous exercise to construct a nonarchimedean absolute value $|\cdot|_2$ on $\mathbb{R}$ such that $|2|_2 < 1$, then show that the partition of $\mathbb{R}^2$ into the subsets
$$\begin{aligned} S_1 &= \{(x,y) \in \mathbb{R}^2 : |x|_2 < 1, |y|_2 < 1\} \\ S_2 &= \{(x,y) \in \mathbb{R}^2 : |x|_2 \geq 1, |x|_2 \geq |y|_2\} \\ S_3 &= \{(x,y) \in \mathbb{R}^2 : |y|_2 \geq 1, |y|_2 > |x|_2\} \end{aligned}$$
   is stable under translation by any $(x,y) \in S_1$.

(b) Prove that no line in $\mathbb{R}^2$ intersects all three subsets. (Hint: reduce to the case where the line passes through $(0, 0)$.)

(c) Suppose we have a dissection of the square $[0, 1] \times [0, 1]$ into triangles. Show that there exists a triangle with one vertex in each of S_1, S_2, S_3. (Hint: assuming there is no such triangle, count segments with one vertex in S_1 and the other in S_2 and find a parity violation. This is closely related to *Sperner's lemma* in combinatorial topology.)

(d) Deduce that if there are m triangles in the dissection and they all have area $1/m$, then $|1/m|_2 > 1$ and therefore m is even.

7. Neukirch, exercise II.6.1: Let K be a field complete with respect to a nonarchimedean absolute value and fix an algebraic closure $\overline{K}$ of K. Let $P(x) \in K[x]$ be a monic polynomial of degree n with roots $\alpha_1, \ldots, \alpha_n \in \overline{K}$ (counted with multiplicity). Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $Q(x)$ is another monic polynomial of degree n with $|P - Q| < \delta$ (for the Gauss norm), then the roots of Q in $\overline{K}$ can be labeled as $\beta_1, \ldots, \beta_n$ in such a way that

$$|\alpha_i - \beta_i| < \epsilon \qquad (i = 1, \ldots, n).$$

That is, "the roots of a polynomial over K vary continuously in the coefficients." (Hint: look at the Newton polygon of $Q(x + \alpha_i)$.)

8. Neukirch, exercise II.6.2 (Krasner's lemma): Let K be a field complete with respect to a nonarchimedean absolute value and fix an algebraic closure $\overline{K}$ of K. Let $\alpha \in \overline{K}$ be separable over K and let $\alpha_1, \ldots, \alpha_n$ be its conjugates over K. If $\beta \in \overline{K}$ is such that

$$|\alpha - \beta| < |\alpha - \alpha_i| \qquad (i = 2, \ldots, n),$$

then $K(\alpha) \subseteq K(\beta)$.