Math 204A (Number Theory), UC San Diego, fall 2022
Problem Set 8 – nominally due December 1, 2022 (but see below)

Due to the academic strike, this problem set (and the previous one) are subject to a *grade on demand* policy. That is, you can let me know (by Zulip PM) at any time before the end of exam week that you have a submission ready for me to grade, and I will record it at the next convenient opportunity.

As a reminder, the maximum possible course grade is 30 out of 40 problems. If you have reached this threshold already, you are encouraged to work out the problems but you do not need to submit anything.

Submit *at most five* of the listed problems.

1. Let K be a finite extension of $\mathbb{Q}_p$.

   (a) Let S be the set of $x \in K$ with the following property: for every positive integer m, there exists a positive integer n_0 (depending on both m and x) such that for each integer $n \geq n_0$, $1 - x^n \in K^{\times m}$. Prove that $S = \{1\} \cap \mathfrak{m}_K$.

   (b) Deduce that every field automorphism of K is continuous. (This improves upon the optional part (c) of PS 6 problem 8.)

2. Let K be the number field $\mathbb{Q}(2^{1/4})$. Let L be the Galois closure of K and put $G = \mathrm{Gal}(L/\mathbb{Q})$.

   (a) Compute the ring of integers $\mathfrak{o}_L$ (e.g., using SageMath).

   (b) Check that there is a single prime $\mathfrak{q}$ of L above 2 and that $\mathfrak{q}$ is totally ramified over 2.

   (c) Compute the groups $G_{\mathfrak{q},s}$ for all s.

   (d) Use the answer to (c) to compute the different of $L/\mathbb{Q}$.

3. Neukirch, exercise II.10.1: Let $K = \mathbb{Q}_p$ and $K_n = K(\zeta)$, where ζ is a primitive p^n-th root of unity. Show that the ramification group of $K_n|K$ are given as follows:

$$\begin{aligned}
G_s &= G(K_n|K) && \text{for } s = 0, \\
G_s &= G(K_n|K_1) && \text{for } 1 \leq s \leq p-1, \\
G_s &= G(K_n|K_2) && \text{for } p \leq s \leq p^2-1, \\
&\cdots \\
G_s &= 1 && \text{for } p^{n-1} \leq s.
\end{aligned}$$

   Note that in this example, the ramification breaks in the upper numbering are integers; this is a special case of the *Hasse–Arf theorem.*

4. Let $G = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ be the quaternion group (i.e., the nonabelian nondihedral group of order 8). Let $Z = \{\pm 1\}$ be the center of G.

(a) Use LMFDB to find a totally ramified extension of local fields L/K for which $G_4 = \{1\}$.

(b) Show that for any extension as in (a), $G = G_0 = G_1$ and $C = G_2 = G_3$.

(c) Deduce that

$$\begin{aligned} G^v &= G \quad \text{for } v \leq 1, \\ G^v &= Z \quad \text{for } 1 < v \leq \tfrac{3}{2}, \\ G^v &= \{1\} \quad \text{for } v > \tfrac{3}{2}. \end{aligned}$$

5. Let K be the field $\mathbb{Q}_p(\pi)$ where $\pi^{p-1} = -p$. Prove that $\zeta_p \in K$. (Hint: use Krasner's lemma or an equivalent argument.)

6. Let L/K be a Galois extension of number fields. Let $\mathfrak{p}$ be a prime of K lying above the prime p of $\mathbb{Q}$. Let $\mathfrak{q}$ be a prime above L. Suppose that $e = e(\mathfrak{q}/\mathfrak{p})$ is not divisible by p. Show that the local different is given by

$$\mathcal{D}_{\mathfrak{o}_{L,\mathfrak{q}}/\mathfrak{o}_{K,\mathfrak{p}}} = \mathfrak{q}^{e-1}.$$

(Hint: first reduce to the case $f(\mathfrak{q}/\mathfrak{p}) = 1$.)

7. Let L/K be a Galois extension of number fields of degree n. Let $\mathfrak{q}$ be a prime ideal of L lying above the prime ideal $\mathfrak{p}$ of K.

(a) Prove that the $\mathfrak{q}$-adic valuation of the different of L/K is at most $e - 1 + v_{\mathfrak{q}}(e)$ where $e = e(\mathfrak{q}/\mathfrak{p})$. (Hint: reduce to the case where $e = [L : K]$, then look at the minimal polynomial of an element of $\mathfrak{q} - \mathfrak{q}^2$. If you get stuck, see Theorem III.2.6 of Neukirch.)

(b) Use (a) to prove that for any fixed positive integers n and D, there are only finitely many number fields of degree n with discriminant $\pm D$. This will be a step in the proof of the Hermite–Minkowski theorem.

8. Use LMFDB to show for cubic number fields which are unramified away from 2 and 3, the upper bound on the absolute discriminant given by the previous exercise is sharp.