Math 203B (Number Theory), UCSD, winter 2015
Notes on completions of fields

Here is a more detailed version of the discussion about completions of fields from the first lecture.

Let F be a field. An *absolutQe value* on F is a function $|\bullet| : F \to [0, +\infty)$ with the following properties.

(a) For $x \in F$, $|x| = 0$ if and only if $x = 0$.

(b) For $x, y \in F$, $|xy| = |x||y|$.

(c) For $x, y \in F$, $|x + y| \leq |x| + |y|$. If in fact we have $|x + y| \leq \max\{|x|, |y|\}$, we say that $|\bullet|$ is a *nonarchimedean* absolute value.

From now on, assume that F comes equipped with a specified absolute value $|\bullet|$. As usual, we say that a sequence $x_1, x_2, \dots$ in F is *Cauchy* if for every $\epsilon > 0$, there exists $N \geq 0$ such that for all integers $m, n \geq N$, we have $|x_m - x_n| < \epsilon$. We record the following observations.

- Any constant sequence is Cauchy.
- Any convergent sequence, and in particular any null sequence (sequence convergent to 0) is Cauchy.
- Any permutation of a Cauchy sequence is Cauchy.
- Any Cauchy sequence is bounded.
- The termwise sum of two Cauchy sequences is Cauchy (by the triangle inequality).
- The termwise product of two Cauchy sequences is Cauchy: if $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ are the two sequences, we see that $(x_1y_1, x_2y_2, \dots)$ is Cauchy by writing

  $$x_m y_m - x_n y_n = x_m(y_m - y_n) + y_n(x_m - x_n)$$

  and using the fact that $|x_m|, |y_n|$ are uniformly bounded. Consequently, the set of Cauchy sequences forms a ring R. By similar reasoning, the termwise product of a bounded sequence and a null sequence is null, so the null sequences form an ideal I in R.

We define the completion $\widehat{F}$ of F, as a ring, as the quotient R/I. Note that the cosets of I in R can be identified with the equivalence classes for the relation $\sim$ defined in class: $(x_1, x_2, \dots) \sim (y_1, y_2, \dots)$ if the sequence $(x_1, y_1, x_2, y_2, \dots)$ is also Cauchy.

We claim that $\widehat{F}$ is not just a ring but a field. To see this, let $(x_1, x_2, \dots)$ be any sequence which is not null. By definition, that means that for some $\epsilon > 0$, there are infinitely many indices n for which $|x_n| > \epsilon$. On the other hand, for some $N \geq 0$, for all $m, n \geq N$, we have $|x_m - x_n| < \epsilon/2$. Consequently, we must have $|x_m| > \epsilon/2$ for all $m \geq N$, so none of these x_m is zero. Let $(y_1, y_2, \dots)$ be any sequence with $y_n = 1/x_n$ for $n \geq N$ (and arbitrary values for $n < N$). For any $\delta > 0$, we can choose $N' \geq N$ such that for $m, n \geq N$, we have $|x_m - x_n| < \delta\epsilon^2/4$; then

$$|y_m - y_n| = \left|\frac{x_n - x_m}{x_m x_n}\right| < \frac{4}{\epsilon^2}|x_n - x_m| < \delta,$$

so $(y_1, y_2, \dots)$ is a Cauchy sequence whose image in $\widehat{F}$ is a multiplicative inverse of $(x_1, x_2, \dots)$.

We define the function $|\bullet|$ on R by taking $|(x_1, x_2, \dots)| = \lim_{n\to\infty} |x_n|$ (which exists by the triangle inequality). One checks easily that this function factors through $\widehat{F}$ and defines an absolute value, and that the map $F \to \widehat{F}$ taking x to $(x, x, \dots)$ is isometric. Moreover, $\widehat{F}$ is complete: if $\underline{x}_1 = (x_{11}, x_{12}, \dots), \underline{x}_2 = (x_{21}, x_{22}, \dots), \dots$ is a sequence in R representing a Cauchy sequence in $\widehat{F}$, we can construct a limit of this sequence by diagonalization. (More precisely, for each i, choose a positive integer N_i such that $|x_{im} - x_{in}| \leq 2^{-i}$ for $m, n \geq N_i$; then one checks that $(x_{1N_1}, x_{2N_2}, \dots)$ belongs to R and represents a limit of the original sequence.)

Now suppose I have constructed some other field G which I would like to show is isomorphic to $\widehat{F}$. It would suffice to produce the following data.

1. An absolute value $|\bullet|$ on G under which it is complete.
2. A homomorphism $F \to G$ of fields (necessarily injective!) which is isometric and whose image is dense.

Namely, given this data, any Cauchy sequence in F maps to a Cauchy sequence in G, which has a limit; we thus get a well-defined homomorphism $\widehat{F} \to G$. This is injective because it's a homomorphism of fields; it's surjective because F is dense in G, so we can represent any element of G as a limit of a Cauchy sequence in F.