Math 204B (Algebraic Number theory), UCSD, winter 2015
Problem Set 1 (due Wednesday, January 14)

Solve the following problems, and turn in the solutions to *four* of them. You may work with other students in the course and/or consult external references, but please indicate *all* such collaborations and consultations. Failure to do so will be considered a violation of academic integrity.

1. Complete the proof sketched in lecture that $\mathbb{Q}_p$, defined algebraically as $\mathbb{Z}_p[1/p]$ where $\mathbb{Z}_p$ is the inverse limit of $\mathbb{Z}/p^n\mathbb{Z}$ over all positive integers n, is also the completion of $\mathbb{Q}$ with respect to the p-adic absolute value, by carrying out the following steps.
   (a) Show that there is a unique ring homomorphism from $\mathbb{Q}$ to $\mathbb{Q}_p$, which is injective.
   (b) Show that $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ with respect to $|\bullet|_p$. Reminder: for $x \in \mathbb{Q}_p$ nonzero, $|x|_p = p^{-v_p(x)}$ where $v_p(x)$ is the unique integer n such that $p^{-n}x$ is a unit in $\mathbb{Z}_p$.
   (c) Show that $\mathbb{Q}_p$ is complete with respect to $|\bullet|_p$.

2. Suppose that $p > 2$. Show that for any $x \in p\mathbb{Z}_p$, the exponential series
$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
and the logarithm series
$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$
converge in $\mathbb{Q}_p$. Then explain what happens when $p = 2$.

3. (a) Show that every field homomorphism $\tau : \mathbb{R} \to \mathbb{R}$ is the identity *without* assuming that τ is continuous. (Hint: use the fact that every nonnegative real number is a perfect square to deduce that τ is order-preserving, then argue that τ must fix $\mathbb{Q}$.)
   (b) Let p be a prime. Prove that the set of $x \in \mathbb{Q}_p$ having an n-th root for all positive integers n not divisible by p is equal to $1 + p\mathbb{Z}_p$. (Hint: use the binomial series, but be careful about the case $p = 2$.)
   (c) Using (b), show that every field homomorphism $\tau : \mathbb{Q}_p \to \mathbb{Q}_p$ is the identity without assuming that τ is continuous.

4. Choose $x \in \mathbb{Z}_p$ not divisible by p. Suppose either that $p > 2$ and x is congruent to a perfect square modulo p, or that $p = 2$ and x is congruent to a perfect square modulo 8. Show that x has a square root in $\mathbb{Z}_p$.

5. Show that for $x, y \in \mathbb{Z}_p$, if $x \equiv y \pmod{p}$ then $x^p \equiv y^p \pmod{p^2}$. Then use this to show that every element of $\mathbb{Z}/p\mathbb{Z}$ lifts uniquely to a solution of the equation $x^p = x$. We will have more to say about these solutions later!

6. Create an account on SageMathCloud (`https://cloud.sagemath.com`), then let me know so that I can add you to the course account. Once you've done that, find the online assignment, do it, and submit it through the web site.