Math 204B (Algebraic Number theory), UCSD, winter 2015
Problem Set 3 (due Wednesday, January 28)

Solve the following problems, and turn in the solutions to *four* of them.

1. Prove that $\mathbb{Q}$ has Haar measure 0 inside $\mathbb{Q}_p$.
2. Neukirch, exercise II.6.1. Hint: look at the Newton polygon of $g(x - \alpha_i)$.
3. Let K be a field complete for a nonarchimedean absolute value. Let L be the completion of an algebraic closure of K for the unique extension of the absolute value. Prove that L is itself algebraically closed. (Hint: use the previous exercise.)
4. (a) Explain how the properties of Newton polygons imply the Eisenstein irreducibility criterion.
   (b) Exhibit an example of a polynomial over $\mathbb{Q}$ which can be shown to be irreducible using Newton polygons over $\mathbb{Q}_p$ for some p, but does not satisfy the Eisenstein criterion for any prime p.
5. Let $\mathfrak{o}$ be a complete discrete valuation ring with residue field k. Let k_0 be a subfield of k which is perfect of characteristic $p > 0$ (so in particular k itself is of characteristic p).
   (a) Show that there is a unique multiplicative (but not additive) map $k_0 \to \mathfrak{o}$ such that the composition $k_0 \to \mathfrak{o} \to k$ coincides with the inclusion $k_0 \to k$. (Hint: for each $x \in k_0$, consider the p^n-th power of a lift of a p^n-th root of x for varying n.)
   (b) Describe the image of the map in (a) in the case $\mathfrak{o} = \mathbb{Z}_p$, $k_0 = k = \mathbb{F}_p$.
   (c) Suppose that $\mathfrak{o}$ has maximal ideal (p) and that k is perfect. Let $\mathfrak{o}'$ be a complete discrete valuation ring with residue field k'. Prove that any homomorphism $k \to k'$ of fields lifts in at most one way to a continuous homomorphism $\mathfrak{o} \to \mathfrak{o}'$. (It turns out that the lift always exists; this can be shown for instance using Witt vectors.)
6. Show that part (a) of the previous exercise fails if we allow k either to be imperfect or to be of characteristic 0.