Math 204B (Algebraic Number theory), UCSD, winter 2015
Problem Set 4 (due Wednesday, February 11)

Solve the following problems, and turn in the solutions to *four* of them. Reminder: no classes or office hours next week (February 2-6); I propose to schedule a make-up lecture on Monday, February 16.

1. Neukirch, Exercise II.7.2.
2. Neukirch, Exercise II.7.3.
3. Let K be a finite abelian extension of $\mathbb{Q}_p$ whose degree is not divisible by p. Prove that $K \subseteq \mathbb{Q}_p(\zeta_n)$ for some positive integer n. (This is the first step in the proof of the *local Kronecker-Weber theorem.*)
4. Here is another step in the proof of the local Kronecker-Weber theorem.
   (a) Prove that if $p > 2$, then there is an extension of $\mathbb{Q}_p$ with Galois group $(\mathbb{Z}/p\mathbb{Z})^n$ for $n = 2$ but not for $n = 3$.
   (b) Prove that there is an extension of $\mathbb{Q}_2$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^n$ for $n = 3$ but not for $n = 4$.
5. Neukirch, exercises II.8.1 and II.8.2 (these count as one problem).
6. Let K be a field of characteristic 0 complete with respect to an absolute value $|\bullet|$ such that $|p| < 1$ for some prime p. Let K' be the inverse limit of K under the p-th power map, as a multiplicative monoid.
   (a) Prove that there is a well-defined operation $+$ on K' given by the formula
   $$(\dots, x_1, x_0) + (\dots, y_1, y_0) = (\dots, z_1, z_0), \qquad z_n = \lim_{n\to\infty} (x_{n+m} + y_{n+m})^{p^m};$$
   the point here is to check that the limit exists.
   (b) Prove that under this operation (and the given multiplication), K' is a field of characteristic p.
7. With notation as in the previous problem, show that the formula $|(\dots, x_1, x_0)|' = |x_0|$ defines an absolute value on K' and that K' is complete.
8. See SageMathCloud for this problem.