Math 204B (Algebraic Number theory), UCSD, winter 2015
Problem Set 5 (due Wednesday, February 18)

Solve the following problems, and turn in the solutions to *four* of them.

1. Exhibit an example of a finite Galois extension K of $\mathbb{Q}$ such that for every prime p, the decomposition group at p is strictly smaller than $\mathrm{Gal}(K/\mathbb{Q})$.

2. Neukirch, exercise II.10.1.

3. Neukirch, exercise II.10.2.

4. With notation as in PS 4, problems 6-7, check that the inverse limits of $\mathfrak{o}_K$ and $\mathfrak{o}_K/(p)$ under the p-th power map coincide. Then calculate the field K' in the following cases:

   (a) $K = \mathbb{Q}_p$;

   (b) K is the completion of $\bigcup_{n=1}^{\infty} \mathbb{Q}_p(\zeta_{p^n})$;

   (c) K is the completion of $\bigcup_{n=1}^{\infty} \mathbb{Q}_p(p^{1/p^n})$.

5. Let K be a field complete with respect to a nonarchimedean absolute value. For d a positive integer, let S_d be the set of monic polynomials of degree d over K, topologized as a vector space over K.

   (a) Prove that for any positive integer d, there exists a nonempty open subset of S_d consisting entirely of reducible polynomials. (Hint: Newton polygons.)

   (b) Prove that for any positive integer d, any irreducible polynomial in S_d admits an open neighborhood consisting entirely of irreducible polynomials. (Hint: remember that the roots of a polynomial vary continuously in the coefficients.)

6. Let K be a field which is complete with respect to two inequivalent nonarchimedean absolute values. Prove that K must be algebraically closed. (Hint: use the previous exercise.)