Math 204B: Number Theory
UCSD, winter 2017
Problem Set 2 (due Wednesday, February 2)

1. Explain how to deduce the usual Chinese remainder theorem for integers from the approximation theorem on $\mathbb{Q}$.

2. Let K be a field which is complete with respect to an absolute value. Let L be an algebraic extension of K which is not finite, but can be written the union of an infinite sequence of finite extensions. Prove that L is not complete.

3. The following constructive form of Hensel's lemma is useful in practice (e.g., for computer calculations). Let F be a field which is complete with respect to a nonarchimedean absolute value. Let $\mathfrak{o}_F$ be the valuation ring of F. Let $P(x) \in \mathfrak{o}_F[x]$ be a (not necessarily monic) polynomial. Suppose that $\alpha \in \mathfrak{o}_F$ satisfies $|P(\alpha)| < |P'(\alpha)|^2$. Set $\alpha_0 = \alpha$ and
$$\alpha_{n+1} = \alpha_n - \frac{P(\alpha_n)}{P'(\alpha_n)} \qquad (n = 0, 1, \dots).$$
Then the sequence α_n converges to a root of F. (Optional: explain what it means to say that this construction is "quadratically convergent.")

4. Let F be a field which is complete with respect to an absolute value. Let n be a positive integer. In this exercise, we consider the statement that "the roots of a degree-n polynomial over F vary continuously in the coefficients."

   (a) Let $\alpha_1, \dots, \alpha_n \in F$ be pairwise distinct elements and write the polynomial $P(T) = (T - \alpha_1) \cdots (T - \alpha_n)$ as $T^n + a_{n-1}T^{n-1} + \cdots + a_0$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $b_0, \dots, b_{n-1} \in F$ satisfy $|b_i - a_i| < \delta$ for all i, then there exist $\beta_1, \dots, \beta_n \in F$ such that $|\alpha_i - \beta_i| < \epsilon$ for $i = 1, \dots, n$. $T^n + b_{n-1}T^{n-1} + \cdots + b_0 = (T - \beta_1) \cdots (T - \beta_n)$

   (b) Prove that (a) can fail if we allow $\alpha_1, \dots, \alpha_n$ not to be pairwise distinct. Hint: you can already find a counterexample with $n = 2$.

5. With notation as in the previous exercise, suppose in addition that F is algebraically closed. Prove that part (a) of that exercise continues to hold if we allow $\alpha_1, \dots, \alpha_n$ not to be pairwise distinct. Now the existence of the roots is clear, and the only issue is to order them so that $|\alpha_i - \beta_i| < \epsilon$ for $i = 1, \dots, n$.