Math 204B: Number Theory
UCSD, winter 2017
Problem Set 3 (due Wednesday, February 15)

1. Let R be the ring of germs at 0 of meromorphic functions in the complex plane, equipped with the exponential valuation given by measuring the order of vanishing at 0. Prove that R is a henselian field which is not complete; what is its completion?

2. Let K be a field of characteristic p which is separably closed and complete with respect to a nonarchimedean absolute value. Prove that K is also algebraically closed. (Hint: you may use the exercise from the previous homework about continuity of the roots of a polynomial with respect to the coefficients.)

3. Prove *Krasner's lemma*: let K be a field complete with respect to a nonarchimedean absolute value $|\bullet|$. Let L be an algebraic closure of K. Let $\alpha_1, \dots, \alpha_n \in L$ be the roots of some polynomial over K. If $\beta \in L$ satisfies

$$|\alpha_1 - \beta| < |\alpha_1 - \alpha_i| \qquad (i = 2, \dots, n),$$

then $K(\alpha) \subseteq K(\beta)$ as subfields of L.

4. Let K be the algebraic closure of a field which is complete with respect to a nonarchimedean absolute value. Prove that the completion of K is again algebraically closed. (Hint: use Krasner's lemma.)

5. Give an example of each of the following. In each case, you need only consider one value of p, and it need not be the same across the cases.

   (a) A reducible polynomial over $\mathbb{Q}_p$ whose Newton polygon is a straight line.

   (b) A polynomial over $\mathbb{Q}_p$ whose Newton polygon has integer slopes, but which nonetheless has no roots in $\mathbb{Q}_p$.

   (c) A polynomial over $\mathbb{Q}_p$ which is irreducible of degree 4 with slope 1/2. (This shows that the denominator of the slope need not equal the degree.)

6. Derive the Eisenstein irreducibility criterion using Newton polygons, then give an example of a new irreducibility criterion that applies in some cases where the Eisenstein criterion fails.