Math 204B: Number Theory
UCSD, winter 2017
Problem Set 4 (due Wednesday, March 1)

1. Let L be the number field $\mathbb{Q}[\alpha]/(\alpha^4 + \alpha^2 + 3)$. Show that the 3-adic valuation v on $\mathbb{Q}$ admits two extensions w_1, w_2 such that $e_{w_1/v} \neq e_{w_2/v}, f_{w_1/v} \neq f_{w_2/v}$.

2. Let L/K be an extension of number fields. Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}_K$ and factor $\mathfrak{p}\mathfrak{o}_L$ into primes $\mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_n^{e_n}$. Prove that the extensions of the $\mathfrak{p}$-adic valuation v on K can be labeled $w_1, \dots, w_n$ in such a way that
$$e_{w_i/v} = e_i, \quad f_{w_i/v} = [\mathfrak{o}_L/\mathfrak{q}_i : \mathfrak{o}_K/\mathfrak{p}] \qquad (i = 1, \dots, n).$$

3. Let K be the field $\mathbb{F}_p((t))$. Let $\overline{K}$ be an algebraic closure of K.
   (a) Show that the maximal unramified subextension of $\overline{K}$ is isomorphic to $\overline{\mathbb{F}}_p((t))$.
   (b) Show that the maximal tamely ramified subextension of $\overline{K}$ is isomorphic to $\bigcup_m \overline{\mathbb{F}}_p((t^{1/m}))$, where m runs over all positive integers not divisible by p.

4. With notation as in the previous exercise, show that $\overline{K}$ is strictly larger than $\bigcup_m \overline{\mathbb{F}}_p((t^{1/m}))$, where m runs over all positive integers (including those divisible by p). Hint: consider the polynomial $P(x) = x^p - x - t^{-1}$.

5. Give, with justification, an example of a finite solvable group G which cannot occur as $\mathrm{Gal}(L/\mathbb{Q}_p)$ for any finite extension L of $\mathbb{Q}_p$.

6. Compute the higher ramification groups of $\mathbb{Q}_p(\zeta_{p^n})$ for p a prime and n a positive integer. (If you need the formula, see the exercises for Neukirch II.10.)

7. In this exercise, we prove the *strong approximation theorem*. Let K be a number field. Let S be a finite set of inequivalent (nontrivial) absolute values of K. Let v_0 be an absolute value inequivalent to each element of S. For each $v \in S$, choose $a_v \in K$. Then for each $\epsilon > 0$, there exists $x \in K$ such that
$$|x - a_v|_v < \epsilon \text{ for each } v \in S$$
$$|x|_v \leq 1 \text{ for each } v \notin S \cup \{v_0\}.$$