

The Hilbert class field

HW 9 to be posted later today, due next Thursday. (The numbering is continued from 204A.)

For those new to 204B, there is a CoCalc project associated to this course. For access to it, see the thread "Announcements/CoCalc" on Zulip. (If you don't know what CoCalc is, watch Thomas Grubb's video from 204A.)

The corresponding section of the CFT notes (2.1) is short enough that I don't expect to spend the full lecture time on it. I will fill in the rest of the time with more details from Chapter 1 (Kummer theory and local Kronecker-Weber); see also the last two lectures of 204A, and also HW 9.

An unramified extension of a number field

$$K = \mathbb{Q}(\sqrt{-5})$$

$$2\mathcal{O}_K = \mathfrak{p}^2$$

$$\mathbb{Q}(\sqrt{-1}, \sqrt{-5}) = L$$

$$\mathbb{Q}(\sqrt{-1}) \quad \mathbb{Q}(\sqrt{-5}) = K$$



$$\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$$

$$\mathfrak{p} = (2, 1 + \sqrt{-5})$$

nonprincipal
 $\mathcal{C}(K) \cong \mathbb{Z}/2\mathbb{Z}$

L/K is unramified over primes
of K lying over odd primes
of \mathbb{Q}

and over \mathfrak{p}

$$L = K(\alpha) \quad \alpha = \frac{1 + \sqrt{5}}{2}$$

has distinct roots mod

has min poly $x^2 - x - 1$
over \mathbb{Q} and K

(i.e. $L = K(\sqrt{5})$ w/ $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$ does not ramify at 2)

Jargon watch: "places"

A place of a number field K is an equivalence class of nontrivial absolute values on K .
Each of these is either a finite place; class of $| \cdot |_p$ for some prime p of \mathbb{Q}_K ,
or infinite place; class of some extension of $| \cdot |_\infty$ on \mathbb{Q} .
nonarchimedean
archimedean
real abs value

Theorem-definition: the Hilbert class field

Let L/K be maximal extension of K which is abelian
and everywhere unramified, every archimedean places
(i.e. real places of K extend to real places of L)
rather than complex places)

Then L/K is finite, and...

L is called the Hilbert class field of K .

e.g. the Hilbert class field of $\mathbb{Q}(\sqrt{-5})$

~~contains~~
's equal to $\mathbb{Q}(\sqrt{-5}, \sqrt{-1})$

e.g. Hilbert class field
of \mathbb{Q} is \mathbb{Q}
(Minkowski:)

The Galois group of the Hilbert class field

... $\text{Gal}(L/K)$ is isomorphic to $Cl(K)$
(in a specific way given by Artin reciprocity)

e.g. $K = \mathbb{Q}(\sqrt{5})$, then $L = \mathbb{Q}(\sqrt{5}, \sigma^{-1})$.

Example: genus theory for quadratic fields

K/\mathbb{Q} quadratic. imaginary if K is real, must be more careful

Gauss: $Cl(K) / 2Cl(K)$ has order 2^{t-1} $\neq \#$ finite places of \mathbb{Q} that ramify in K .

Via theorem, this means that K has an unramified extension which is Galois with Galois group $(\mathbb{Z}/2\mathbb{Z})^{t-1}$

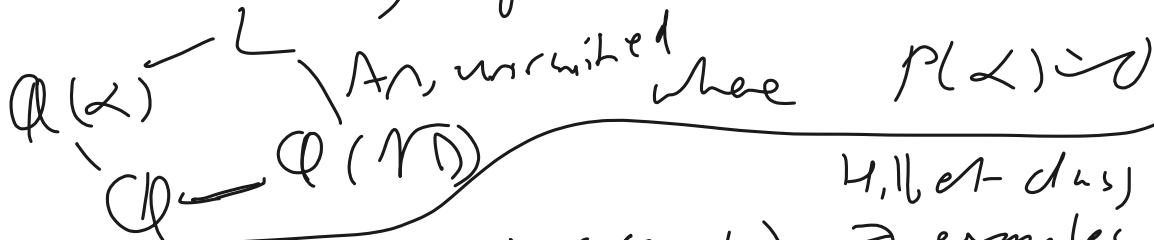
This can be seen explicitly (exercise)

Unramified extensions can be nonabelian

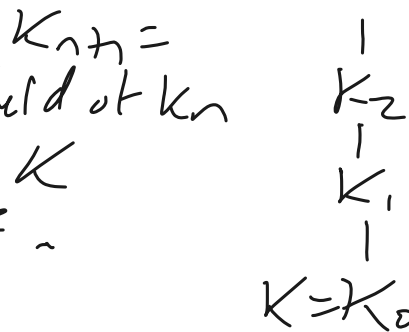
e.g. if $P(x) \in \mathbb{R}[x]$ has square free discriminant D & irreducible with Galois group S_n

$K = \mathbb{Q}(\sqrt{D})$. Then K_1 has an unramified extension with Galois group A_n .

Galois group S_n
 $n = \deg(P)$



Class field
 tower:



M. M. Golod-Shafarevich: \Rightarrow examples of K for which the class field tower is infinite - explicit

Now, back to Kummer theory

Thm (Kummer): K field containing ζ_n (primitive n -th root of unity)

The $\mathbb{Z}/n\mathbb{Z}$ -extensions of K are of form

$K(\alpha^{1/n})$ where $\alpha \in K^\times$ represents an element of $K^\times / (K^\times)^n$ of order n .

pf let L/K be a $\mathbb{Z}/n\mathbb{Z}$ -extension. let $g \in \text{Gal}(L/K)$ send $\alpha^{1/n}$ to $\zeta_n^r \alpha^{1/n}$.

This defines an element of $H^1(\text{Gal}(L/K), L^\times)$, i.e. f satisfies

$$f(hg) = h(f(g)) f(h) \quad \forall s, h \in \text{Gal}(L/K).$$

(crossed
2-cocycle)

Hilbert: any crossed homomorphism has form

$$f(h) = h(b)/t \quad \text{for some } t \in L^\times. \Rightarrow t = \alpha^{1/n} \text{ for some } \alpha \in K^\times.$$

Kummer theory and local Kronecker-Weber

Recall that to prove local KW, we need
to show by \mathbb{Z}_p^\times -extensions of \mathbb{Q}_p .
($p > 2$)

to do this, first look at $\mathbb{Q}_p(\zeta_p)$ $\pi = p-1$

$$\begin{aligned} \underline{\mathbb{Q}_p(\zeta_p)^*} &= \pi \mathbb{Z} \times \mathbb{Z}_p^\times [\zeta_p]^* \\ &= \pi \mathbb{Z} \times \langle \zeta_{p-1} \rangle \times U_1 \quad \begin{matrix} x \equiv 1 \\ \text{mod } \pi \end{matrix} \end{aligned}$$

$$\underline{(\mathbb{Q}_p(\zeta_p)^*)^n} = \pi^n \mathbb{Z} \times \langle \zeta_{p-1} \rangle \times \underbrace{U_1^{\otimes n}}_{= U_{p+n}} \quad \begin{matrix} x \equiv 1 \\ \text{mod } \pi^{p+n} \end{matrix} \quad (\text{HW})$$