Generalized ideal class groups and Artin reciprocity

HW 9 has been posted, due Thursday, January 14. Again, please let me know in advance of the deadline if you need an extension.

As with Math 204A, non-UCSD students can take this course for credit via concurrent enrollment (for a fee which I do not control). Please let me know via PM on Zulip if you are planning to do this.

Land acknowledgment: UC San Diego's main campus is located near the Pacific Ocean on approximately 1,200 acres of coastal woodland in La Jolla, California. The campus sits on the ancestral homelands of the Kumeyaay Nation. Kumeyaay people continue to have an important and thriving presence in the region. [Additional note: the name "La Jolla" is of Kumeyaay origin.]
$K = \mathbb{Q}(\sqrt{-5})$  
$L = \mathbb{Q}(\sqrt{-5}, \sqrt{-7})$

$L/K$ is an everywhere unramified abelian extension.

$\text{real}(L/K) \subseteq \mathbb{Z}/2 \mathbb{Z} \cong C_1(K)$

Then a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ splits in $L$ if and only if it is principal. (In other words, $\mathfrak{p} \mathcal{O}_L = \mathfrak{a}_1 \cdots \mathfrak{a}_g$ for some above $\mathfrak{p}$ in $C_1(K)$.)
Splitting of primes in this example

- If \( p = (\beta) \) where \( p \neq 2,3 \) is a rational prime, which is inert in \( K \).

\[
\left( \frac{-5}{\mathfrak{p}} \right) = -1 \Rightarrow \text{either } \left( \frac{-1}{\mathfrak{p}} \right) = 1 \text{ or } \left( \frac{5}{\mathfrak{p}} \right) = 1
\]

\[
\Rightarrow \mathcal{O}_L = \mathcal{O}_K(\sqrt{-5}) = \mathcal{O}_K(\sqrt{5})
\]

- If \( \mathfrak{p} = (p) \) unless \( p \neq 2,3 \) splits in \( K \).

- If \( \mathfrak{p} = (\beta) \) then \( \beta = x + \sqrt{5}y \) implies \( x^2 + 5y^2 = \mathfrak{p} \).

\[
\Rightarrow \mathfrak{p} = (2m + \sqrt{5y})
\]

- If \( \mathfrak{p} = (2,1+\sqrt{-5}) \), taking norms \( \Rightarrow \mathfrak{p} = 2(1+\sqrt{-5}) \).

\[
\Rightarrow \mathfrak{p} = 3 \text{ mod } 4
\]

\[
\Rightarrow \mathfrak{p} \text{ does not split in } L.
\]
Compare this with Kronecker-Weber

\( K/\mathbb{Q} \, \text{abelian but typically ramified} \)

by \( \mathcal{K} \sim \mathcal{O}_n \) for some \( n \)

primitive \( n \)-th root of 1

\( p \) prime \( \implies \) splitting \( \mathfrak{p} \) at \( (p) \) in \( \mathcal{O}_n \)

is determined by \( r \) mod \( n \).

(relates to)

\[ \left\lfloor \frac{r}{n} \right\rfloor \mathcal{O}_n \]

\( \mathcal{F}_p(\mathcal{O}_n) \)

\( \mathcal{F}_p - \# \mathcal{F}_p(\mathcal{O}_n) = p^a \)

where \( a \) is least integer \( \leq \frac{r}{n} \).

\( p^a \equiv 1 \mod n \).
Generalized ideal class groups

\[ \mathcal{I}_K = \{ \text{fractional ideals of } K \text{ coprime to } m \} \]

\[ \mathcal{P}_K = \{ \text{fractional ideals of } K \text{ of form } (c) \text{ where } c \in \mathbb{F}_p \} \]

The quotient \[ \mathcal{I}_K / \mathcal{P}_K \cong C_1^m(\mathcal{O}_K) \]

is the generalized ideal class group of \( K \) or \( \mathcal{O}_K \) modulo \( m \).

\[ \text{Field } K = \text{ field} \]

\[ m = \text{ product of places of } K \]

\[ \text{normalized } \mathcal{O}_K = \text{ formal product of infinite places of } K \times \text{ formal product of finite places} \]
The Artin map

Let $f$ be a prime of $OK$.

Because $f$ does not lie in $L$, decomposition of $\mathfrak{F}_p$ is cyclic, generated by $G$. $G$ is cyclic generated by $G'$.

Because $\mathfrak{F}_p$ abelian, $G'$ depends only on $f$, $\mathfrak{F}_p$ does not lie in $\mathfrak{F}_p = \mathfrak{F}_p + \mathfrak{F}_p$.

By multiplication, set $I/K \rightarrow \text{Gal}(L/K)$ for some normal subgroup.
Artin reciprocity

The Artin reciprocity of LK is the

The conclusion of LK is the

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Ray class fields

For a ray class field $K$ of $\mathbb{Q}$, a ray class field $K$ of $\mathbb{Q}$ is a Galois
extension of $\mathbb{Q}$ such that $\text{Cl}_{\mathbb{Q}}(n) \to \text{Gal}(K/\mathbb{Q})$
exists and is a bijection.

- If $m = 1$, this is the Hilbert class field $\mathbb{Q}(\zeta_n)$.

- If $K = \mathbb{Q}$, $m = 1$, this is the $\mathbb{Q}$-class field $\mathbb{Q}(\zeta_n)$.

- If $K = \mathbb{Q}$, $m = \infty$, this is the $\mathbb{Q}$-class field $\mathbb{Q}(\zeta_\infty)$.

Note: The Hilbert symbol implies uniqueness of ray class fields.
The existence theorem

For every number field $K$, there exists a ray class field $L/K$.

Meanwhile, Artin reciprocity tells us that every finite abelian extension of $K$ is contained in some ray class field $L/K$.

The union of all abelian extensions of $K$ is the union of all ray class fields.
Inexpliciteness of the existence theorem

For most $K$ the existence theorem is not constructive!

$K = \mathbb{Q}(\sqrt{-1})$ Kronecker-Weber (torsion of $\mathbb{Q}_m$)

$K = \mathbb{Q}(\sqrt{-1})$ CM elliptic curves

function fields, printed modules

$K = \text{finite extension of } \mathbb{Q} - \text{Lichtenbaum}$

$\mathfrak{m}$ (Shimura)

??  (Beukers-Dwork)
Bonus observation: the principal ideal theorem

\[ K = \mathbb{Q}(\sqrt{5}) \quad \mathcal{O} = \mathbb{Z}[\sqrt{5}, \sqrt{7}] \]

Note: \((2, 1 + \sqrt{5})\mathcal{O}_K = (1 + \sqrt{5}) \mathcal{O}_K.

\Rightarrow \quad \text{Every } \mathfrak{p} \subseteq \mathcal{O}_K \text{ of norm } 2 \text{ is principal.}