Zeta functions and the Chebotarëv density theorem

Schedule adjustment: no lecture or office hours on Friday, January 15. Instead, that day's lecture will be given (and recorded) Thursday, January 14 at 4pm, and followed by 30 minutes of office hours.

Additional reminder: no lecture or office hours on Monday, January 18 (university holiday).

HW 9 is due Thursday, January 14. Please ask *in advance* for any extensions on homework. (Enrolled students only need to complete 6 out of 9 problem sets for full credit.)

HW 10 will be posted later today. It will be due Thursday, January 21.

The Dedekind zeta function of a number field

K = # field. <u>Dedekind zeta function</u>

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - \mathrm{Norm}(\mathfrak{p})^{-s})^{-1}$$

$\mathfrak{p}$ nonzero prime of K

absolutely convergent For $\mathrm{Re}(s) > 1$.

via unique factorization of ideals

$$= \sum_{\mathfrak{a}} \mathrm{Norm}(\mathfrak{a})^{-s}$$

$\mathfrak{a}$ nonzero ideal of $\mathcal{O}_K$

for $K = \mathbb{Q}$

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$$

<u>Thm</u> $\zeta_K(s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s=1$, no other poles.

(Residue at $s=1$ appears in <u>class number formula</u>.)

L-functions of abelian characters

K ff field

$\underline{m}$ = formal product of places of K

$\chi_{\underline{m}} : Cl^{\underline{m}}(K) \longrightarrow \mathbb{C}^{\times}$ an abelian character

$I_K^{\underline{m}}$ ray class group

say $\chi_{\underline{m}}(\underline{a}) = 0$ if $\underline{a}$ not coprime to $\underline{m}$

$$L(\chi_{\underline{m}}, s) = \prod_{p} \left(1 - \chi_{\underline{m}}(p) \operatorname{Norm}(p)^{-s}\right)^{-1}$$

$$= \sum_{\underline{a}} \chi_{\underline{m}}(\underline{a}) \operatorname{Norm}(\underline{a})^{-s}$$

Analytic properties of L-functions

If $\chi_{\underline{m}}$ is trivial, then $L(\chi_{\underline{m}}, s) = \zeta_K(s) \times$ finite # of factors

Thm If $\chi_{\underline{m}}$ is nontrivial, then $L(\chi_{\underline{m}}, s)$ converges absolutely on $\mathrm{Re}(s) > 1$ and extends to a holomorphic function on $\mathbb{C}$.

Thm If χ_m is nontrivial, then $L(\chi_m, 1) \neq 0$.
(implies that $\log L(\chi_m, s)$ is holomorphic at $s = 1$.)

Dirichlet density

S = set of primes in K

S has Dirichlet density $d \in [0,1]$ if

$$\lim_{s \to 1^+} \frac{\sum_{p \in S} \mathrm{Norm}(p)^{-s}}{\log \frac{1}{s-1}} = d$$

previous theorems imply: $\underline{m}$ = formal product of places

For each element of $Cl^m(K)$, set of prime ideals p which land in this class has Dirichlet density $= 1/\#Cl^m(K)$

Dirichlet density vs. natural density

More natural notion of density is

$$\lim_{x\to\infty} \frac{\#\{p : p \in S,\ \mathrm{Norm}(p)\le x\}}{\#\{p : \mathrm{Norm}(p)\le x\}}$$

"only count primes of absolute degree 1"

Existence of natural density $\Rightarrow$ Dirichlet density but not vice versa.

(In all statements from today, also hold for natural density, but with more work. Under GRH, also hold with effective power-saving error terms.)

Statement of the Chebotarëv density theorem

L/K Galois extension of # fields,

For each prime $\mathfrak{p}$ of K which does not ramify in L,

for each prime $\mathfrak{q}$ of L above $\mathfrak{p}$, get decomposition group $G_{\mathfrak{q}} \ni \mathrm{Frob}_{\mathfrak{q}}$. Conjugacy class of $\mathrm{Frob}_{\mathfrak{q}}$ depends only on $\mathfrak{p}$.

$G_{\mathfrak{q}} \subset G = \mathrm{Gal}(L/K)$

<u>Thm</u> For each conjugacy class C in G, the set of primes $\mathfrak{p}$ of K of abs. degree 1 whose Frobenius class is C has Dirichlet / natural density $= \dfrac{\#C}{\#G}$.

A corollary

Let L/K be nontrivial extension (not necess. Galois) of # fields. Then $\exists$ infinitely many prime ideals of K which do not split completely in L.

Pf Let M/K Galois closure of L/K. Then $\mathfrak{p} \subset K$ splits completely in $L \Leftrightarrow$ splits completely in M $\Leftrightarrow$ Frob conjugacy class of $\mathfrak{p}$ for M/K is trivial class of $\mathrm{Gal}(M/K)$

Proof of the Chebotarëv density theorem

- For L/K abelian, Artin reciprocity equates this statement with the previous statement of equidistribution in $Cl^m(K)$
- For general L/K: pick $g \in G = \mathrm{Gal}(L/K)$ $(G \supseteq Z)$

$Z = Z(g)$ centralizer: class of g has size $\frac{\#G}{\#Z}$

Let f = order of g $\quad K'$ = fixed field of g.

For each prime of K of abs degree 1 with Frobenius in class of g,

there are $\#Z/f$ primes of K' above it (also of abs degree 1)

with Frob g …

L

cyclic abelian

K' ← decomposition field

K

Additional remarks

There are many more equidistribution statements/conjectures in # theory (e.g. Sato-Tate conjecture)

Many of these statements admit a common generalization (Serre).

lectures on $N_X(p)$