Zeta functions and the Chebotarëv density theorem

Schedule adjustment: no lecture or office hours on Friday, January 15. Instead, that day's lecture will be given (and recorded) Thursday, January 14 at 4pm, and followed by 30 minutes of office hours.

Additional reminder: no lecture or office hours on Monday, January 18 (university holiday).

HW 9 is due Thursday, January 14. Please ask in advance for any extensions on homework. (Enrolled students only need to complete 6 out of 9 problem sets for full credit.)

HW 10 will be posted later today. It will be due Thursday, January 21.
The Dedekind zeta function of a number field

\[ \zeta_K(s) = \prod_{\mathfrak{p}} \left( 1 - \operatorname{Num}(\mathfrak{p})^{-s} \right)^{-1} \]

For \( \Re(s) > 1 \), \( \zeta_K(s) \) is absolutely convergent.

\[ \frac{\zeta_K(s)}{\zeta(s)} = \prod_{\mathfrak{p}} \left( 1 - \nu_{\mathfrak{p}}^{-s} \right) \]

Fix \( K = \mathbb{Q} \).

\[ \zeta(s) = \prod_{n=1}^{\infty} \frac{1}{1 - n^{-s}} \]

The \( \zeta_K(s) \) extends to a meromorphic function on \( \mathbb{C} \) with a simple pole at \( s = 1 \), no other pole.

(Residue at \( s = 1 \) appears in class number formula.)
L-functions of abelian characters

\[ L(K_m, s) = \prod_{\mathfrak{p}} \left( 1 - \frac{\chi_{m}(\mathfrak{p}) \text{Norm}(\mathfrak{p})^{-s}}{\mathfrak{p}^s} \right)^{-1} \]

with \( \chi_{m} \) being the \( m \)-th power of a character of \( \text{Cl}(K) \) 

\( \chi_{m} \) is a character of \( \text{Cl}(K) \) mapping \( \mathfrak{p} \) to \( \chi(\mathfrak{p}) \), which is a character of \( \text{Cl}(K) \), \( \mathfrak{p} \) being a place of \( K \).

For every class group \( \mathcal{O}_K \), we have \( \chi_{m}(a \mathcal{O}_K) = 0 \) if \( a \) is not congruent to \( m \mod \mathfrak{m} \).
Analytic properties of L-functions

If \( \chi_m \) is trivial, then \( L(\chi_m, s) = \zeta_k(s) \) x \( \prod_{\text{finite}} \)

If \( \chi_m \) is non-trivial, then \( L(\chi_m, s) \) converges absolutely for \( \Re(s) > 1 \) and extends to a holomorphic function on \( \mathbb{C} \).

If \( \chi_m \) is non-trivial, then \( L(\chi_m, 1) \neq 0 \).

(implies that \( \log L(\chi_m, s) \) is holomorphic \( \Re(s) = 1 \)).
Dirichlet density

$S = \text{set of primes in } K$

$S$ has Dirichlet density $d \in (0, 1)$ if

\[
\lim_{s \to 1^+} \frac{1}{\log \frac{1}{s-1}} \sum_{p \in S} \frac{1}{\nu_p(\nu_s(p)) - 5} = d
\]

provides power simply: $m = \text{formal product of}$ since for each element of $\text{Cl}^m(K)$, set of some ideals of which lands in this class has Dirichlet density $1 / \# \text{Cl}^m(K)$
Dirichlet density vs. natural density

More natural density is

$$\# \{ \mathbf{p} : \mathbf{p} \in S, \text{Norm}(\mathbf{p}) \leq x \}$$

More

Existence of natural density $\Rightarrow$ Dirichlet density

but not vice versa.

In all statements from today, also hold for natural density, but with more work.

Under GRH, also hold with effective zero-sieving error terms.
Statement of the Chebotarëv density theorem

Let \( L/K \) be an extension of \# fields. For each \( \mathfrak{p} \) of \( K \) which does not ramify in \( L \), for each prime \( \mathfrak{q} \) at \( L \) above \( \mathfrak{p} \), set the decomposition

\[
\sum_{\mathfrak{q} \supset \mathfrak{p}} \frac{1}{\# \mathfrak{q}} = \frac{1}{\# \mathfrak{p}}. \]

Conjugacy classes depend solely on \( \mathfrak{p} \).

Then for each conjugacy class \( C \) in \( G \), the set of primes \( \mathfrak{p} \) of \( K \) whose Galois classes is \( C \) has Dirichlet density \( \frac{\# C}{\# G} \).
A corollary

\[
\text{Let } L/K \text{ be a normal extension (not necessarily complete, not necessarily algebraic). Then there exists a finite set of elements in } K \text{ which do not split completely in } L.
\]

If \( L/K \) is an algebraic extension of \( K \), then \( \mathfrak{p} \subset K \) splits completely in \( L \) if and only if \( \mathfrak{p}(L/K) \) splits completely in \( L \).

\((=)\) For any conjugacy class of \( \mathfrak{p} \) in \( \text{Gal}(L/K) \), the class of \( \mathfrak{p}(L/K) \) in \( \text{Gal}(L/K) \).
Proof of the Chebotarëv density theorem

For $G$ abelian, Artin reciprocity equates this statement with the previous statement of equivalence when $G = \text{Gal}(K'/\mathbb{Q})$.

For several $L/K$, with $G = \text{Gal}(L/K)$, let $C = \mathbb{Z}/2\mathbb{Z}$ be the class of $C$ has size $\frac{|G|}{|C|}$. Let $f$ be any of $5$ classes of $G$. For each prime $P$ of $K$ of abeditude $\ell$ and $G = \text{Gal}(L/K)$, there are $\#2/5$ primes of $K'$ above $P$ (also of abeditude $\ell$) which are a decomposition field.
Additional remarks

There are many more equidistribution statements/conjectures in $\mathbb{F}_p$ theory (e.g. Suba-Tate conjecture $C$).

Many of these statements admit a common generalization (Serre) in lectures on $M_\mathbb{F}(p)$.